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Index Ellipsoid Analysis (*October 18, 2019*)†

Problem Statement

Assume an anisotropic, linear, homogeneous, and non-magnetic medium for which the Maxwell's equations for plane-wave solutions are:

$$\begin{aligned}
\vec{k}\times\vec{H} &= -\omega\vec{D} = -\omega[\epsilon]\vec{E} = -\omega\epsilon_0[\varepsilon]\vec{E},\\
\vec{k}\times\vec{E} &= \omega\vec{B} = \omega\mu_0\vec{H},\\
\vec{k}\cdot\vec{D} &= \vec{k}\cdot[\epsilon]\vec{E} = 0,\\
\vec{k}\cdot\vec{B} &= \vec{k}\cdot\mu_0\vec{H} = 0,
\end{aligned}$$

where $\vec{k}$ is a wavevector of the form $\vec{k} = k_0 n\hat{k} = k_0 n(a_x\hat{\imath}_x + a_y\hat{\imath}_y + a_z\hat{\imath}_z)$ with k_0 the freespace wavenumber, n the corresponding refractive index for the direction of propagation of this wavevector, and $\hat{k}$ the unit vector along the direction of propagation (with a_x, a_y, and a_z its directional cosines). The matrix (tensor) $[\epsilon] = \epsilon_0[\varepsilon]$ is the permittivity matrix (with $[\varepsilon]$ the relative permittivity matrix). Assume that $[\mathcal{A}] = [\varepsilon]^{-1}$ is the inverse relative permittivity tensor (impermeability tensor). For numerical implementation the following parameters are used: $n_{xx} = 1.552$, $n_{yy} = 1.582$, $n_{zz} = 1.588$, and $a_x = a_y = a_z = 1/\sqrt{3}$.

(a) Show that the displacement vector $\vec{D}$ satisfies the following eigenvalue/eigenvector equation

$$\hat{k}\times\left[\hat{k}\times\left([\mathcal{A}]\vec{D}\right)\right] + \frac{1}{n^2}\vec{D} = 0.$$

Solve this equation in the principal axes system xyz shown in Fig. 1.

(b) Since $\vec{D}$ is always perpendicular to $\vec{k}$ it is convenient to use a new coordinate system with one axis coinciding with the direction of propagation of the wave (assume the unit vector in this direction as $\hat{\imath}_3 = \hat{k}$). Then denote the two new transverse axes by 1 and 2 with unit vectors $\hat{\imath}_1$ and $\hat{\imath}_2$ respectively. In this new coordinate system $\hat{k} = [0\ 0\ 1]^T = \hat{\imath}_3$. Furthermore, assume that $\vec{D}$ and $[\mathcal{A}]$ are now expressed in this new coordinate system. Since $\vec{k}\cdot\vec{D} = 0$, $\vec{D} = [D_1\ D_2\ 0]^T = D_1\hat{\imath}_1 + D_2\hat{\imath}_2$. Show that the previous equation can be written as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\vec{D}_t = [\mathcal{A}_t]\vec{D}_t = \frac{1}{n^2}\vec{D}_t,$$

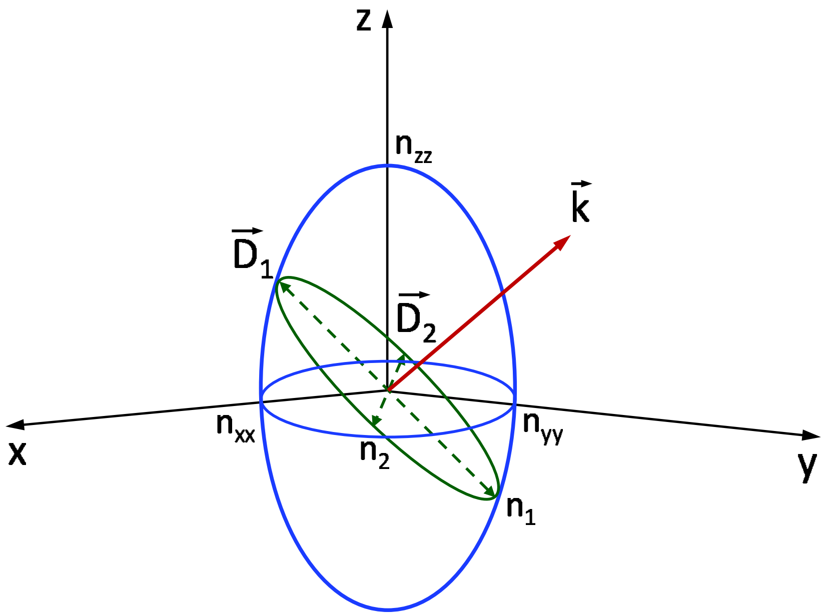

Figure 1: The index ellipsoid expressed in the principal axes system. The wavevector $\vec{k}$ denotes the direction of propagation. The intersection of the normal to the wavevector plane, at the origin, with the index ellipsoid specifies an ellipse (green in the figure). The resulting two eigen-polarizations (directions shown by $\vec{D}_1$ and $\vec{D}_2$), and the two-eigen indices are specified by the semi-axes of that ellipse. The n_{ww} ($w = x, y, z$) are the principal refractive indices.

where $[\mathcal{A}_t]$ is the transverse impermeability tensor (symmetric) and $\vec{D}_t = [D_1 \;\; D_2]^T$. The above equation is again an eigenvalue/eigenvector equation in two dimensions. The two eigenvalues and the two eigenvectors correspond to the two refractive indices for the direction of propagation and their corresponding eigen-polarizations. Find the two eigen-indices and the two eigen-polarizations. Show that the two eigen-polarizations are orthogonal to each other.

(c) Now show that the ellipse which is defined from the index ellipsoid as an intersection of the index ellipsoid with a plane perpendicular to $\hat{k}$ and passing through the origin has semi-axes with lengths equal to the squares of the eigen-indices. In addition, show that the directions of the two semi-axes correspond to the two eigen-polarizations.

Solution

(a) From Maxwell's equations for plane wave solutions the following steps can be easily written:

$$
\begin{aligned}
\vec{k}\times\vec{H} &= -\omega\vec{D} \Longrightarrow \\
\vec{k}\times\left(\frac{1}{\omega\mu_0}\vec{k}\times\vec{E}\right) &= -\omega\vec{D} \Longrightarrow \\
\vec{k}\times\left(\vec{k}\times\frac{1}{\epsilon_0}[\varepsilon]^{-1}\vec{D}\right) &= -\omega^2\mu_0\vec{D} \Longrightarrow \\
\vec{k}\times\left(\vec{k}\times[\mathcal{A}]\vec{D}\right) &= -\omega^2\epsilon_0\mu_0\vec{D} = -k_0^2\vec{D} \Longrightarrow \\
\hat{k}\times\left(\hat{k}\times[\mathcal{A}]\vec{D}\right) &= -\frac{1}{n^2}\vec{D}. \qquad (1)
\end{aligned}
$$

At this point let's attempt to solve the latter equation in the principal axis system xyz. Using the vector identity $(\vec{a}\times(\vec{b}\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$, in Eq. (1), it can be written as:

$$
\left[\hat{k}\cdot[\mathcal{A}]\vec{D}\right]\hat{k} - \underbrace{\left[\hat{k}\cdot\hat{k}\right]}_{=1}[\mathcal{A}]\vec{D} = -\frac{1}{n^2}\vec{D}
$$

$$
\begin{bmatrix}
\frac{1-a_x^2}{n_{xx}^2} & -\frac{a_xa_y}{n_{yy}^2} & -\frac{a_xa_z}{n_{zz}^2} \\
-\frac{a_ya_x}{n_{xx}^2} & \frac{1-a_y^2}{n_{yy}^2} & -\frac{a_ya_z}{n_{zz}^2} \\
-\frac{a_za_x}{n_{xx}^2} & -\frac{a_za_y}{n_{yy}^2} & \frac{1-a_z^2}{n_{zz}^2}
\end{bmatrix}
\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \frac{1}{n^2}\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}. \qquad (2)
$$

In the last derivation $[\mathcal{A}] = diag[1/n_{ww}]$ (where $w = x, y, z$). It is straightforward to show that the determinant of the 3×3 matrix of Eq. (2) is equal to $-(a_x^2 + a_y^2 + a_z^2 - 1)/(n_xx^2n_{yy}^2n_{zz}^2) = 0$ since always $a_x^2 + a_y^2 + a_z^2 = 1$. Therefore, one of the eigenvalues of Eq. (2) is equal to zero. The remaining two eigenvalues correspond to the $1/n^2$ inverse eigen-indices while the corresponding eigenvectors correspond to the two polarizations. Using the data for the numerical implementation, the fast and slow indices and the corresponding polarizations are:

$$
n_f = 1.56264128 \qquad \hat{D}_f = [-0.81416, +0.46057, +0.35359]^T, \qquad (3)
$$

$$
n_s = 1.58512289 \qquad \hat{D}_s = [-0.06176, -0.67420, +0.73596]^T, \qquad (4)
$$

Since the matrix involved is of only third order and since its determinant is always zero due to the directional cosines involved the characteristic polynomial of the matrix of Eq. (2) becomes (calculations where performed in *Wolfram Mathematica*):

$$
\begin{aligned}
F(\rho) &= \rho\left[\rho^2 - A\rho - B\right] = 0, \\
A &= \frac{n_{xx}^2n_{yy}^2 + n_{xx}^2n_{zz}^2 - a_z^2n_{xx}^2n_{yy}^2 - a_y^2n_{xx}^2n_{zz}^2 + a_y^2n_{yy}^2n_{zz}^2 + a_z^2n_{yy}^2n_{zz}^2}{n_{xx}^2n_{yy}^2n_{zz}^2}, \\
B &= \frac{-n_{xx}^2 + a_y^2n_{xx}^2 + a_z^2n_{xx}^2 - a_y^2n_{yy}^2 - a_z^2n_{zz}^2}{n_{xx}^2n_{yy}^2n_{zz}^2} = -\frac{a_x^2n_{xx}^2 + a_y^2n_{yy}^2 + a_z^2n_{zz}^2}{n_{xx}^2n_{yy}^2n_{zz}^2},
\end{aligned}
$$

and the resulting eigenvalues are:

$$\begin{aligned}
\frac{1}{n_1^2} &= \rho_1 = \frac{1}{2}\left[A + \sqrt{A^2+4B}\right], \\
\frac{1}{n_2^2} &= \rho_2 = \frac{1}{2}\left[A - \sqrt{A^2+4B}\right], \\
\rho_3 &= 0.
\end{aligned} \tag{5}$$

The refractive indices n_1 and n_2 correspond to the eigen-indices of the two corresponding eigen-polarizations. Of course the zero eigenvalue does not have any physical meaning and it is neglected. The eigenvectors can be also determined analytically since the eigenvalues have been found. Specifically, it can be shown that if $n_f = min(n_1, n_2)$ and $n_s = max(n_1, n_2)$ then the two eigen-polarizations are:

$$\begin{aligned}
\hat{D}_w &= [D_{xw}, D_{yw}, D_{zw}]^T \frac{1}{\sqrt{D_{xw}^2 + D_{yw}^2 + D_{zw}^2}}, \qquad w = s, f, \quad \text{and} \\
D_{xw} &= \frac{D_{zw} a_z}{\mathscr{D}(\rho_w) n_{zz}^2}\left[a_{22}(\rho_w) a_x - a_{12}(\rho_w) a_y\right], \\
D_{yw} &= \frac{D_{zw} a_z}{\mathscr{D}(\rho_w) n_{zz}^2}\left[-a_{12}(\rho_w) a_x + a_{11}(\rho_w) a_y\right], \\
D_{zw} &= 1, \\
\mathscr{D}(\rho_w) &= a_{11}(\rho_w) a_{22}(\rho_w) - a_{12}(\rho_w) a_{21}(\rho_w), \\
a_{11}(\rho_w) &= \frac{1 - a_x^2}{n_{xx}^2} - \rho_w, \\
a_{12}(\rho_w) &= -\frac{a_x a_y}{n_{yy}^2}, \\
a_{21}(\rho_w) &= -\frac{a_x a_y}{n_{xx}^2}, \\
a_{22}(\rho_w) &= \frac{1 - a_y^2}{n_{yy}^2} - \rho_w.
\end{aligned}$$

The numerical application of the last equations in the case of $n_{xx} = 1.552$, $n_{yy} = 1.582$, $n_{zz} = 1.588$, and $a_x = a_y = a_z = 1/\sqrt{3}$, can produce practically the same numerical results found earlier using numerical evaluation of the eigenvalues/eigenvectors problem. It is worth mentioning, that the numerical evaluation of eigenvalues/eigenvectors seems more accurate than the analytical counterpart presented previously. This can be determined when examining how perpendicular the two eigen-polarizations are. Specifically the dot product of the eigen-polarizations determined with a numerical technique in Eqs. (3) and (4) is of the order of 10^{-15} while the corresponding dot product of the analytically computed eigenvectors is of the order of 10^{-13}.

(b) Let's determine the unit vectors $\hat{\imath}_1$ and $\hat{\imath}_2$ given that $\hat{\imath}_3 = \hat{k}$. The selection of $\hat{\imath}_1$ and $\hat{\imath}_2$ is arbitrary since they can be any two perpendicular to each other and perpendicular to $\hat{\imath}_3$ unit vectors. The $\hat{\imath}_2$ can be defined as $\hat{\imath}_2 = \hat{\imath}_3 \times \hat{\imath}_z / |\hat{\imath}_3 \times \hat{\imath}_z|$ (in the case that $\hat{\imath}_3 = \hat{k} \neq \hat{\imath}_z$). Then, $\hat{\imath}_1 = \hat{\imath}_2 \times \hat{\imath}_3 / |\hat{\imath}_2 \times \hat{\imath}_3|$. Then, the three new unit vectors in the 123 coordinate system can be expressed as follows (for $a_z \neq \pm 1$):

$$\begin{aligned}
\hat{\imath}_3 &= a_x \hat{\imath}_x + a_y \hat{\imath}_y + a_z \hat{\imath}_z, \\
\hat{\imath}_2 &= \frac{1}{\sqrt{1-a_z^2}} \left[a_y \hat{\imath}_x - a_x \hat{\imath}_y \right], \\
\hat{\imath}_1 &= \frac{1}{\sqrt{1-a_z^2}} \left[-a_x a_z \hat{\imath}_x - a_y a_z \hat{\imath}_y + (a_x^2 + a_y^2) \hat{\imath}_z \right].
\end{aligned}$$

In the case that $a_z = \pm 1$ then $\hat{\imath}_3 = \hat{\imath}_z$, $\hat{\imath}_2 = \hat{\imath}_y$, and $\hat{\imath}_1 = \hat{\imath}_x$ and this is a trivial case since it is propagation along one of the principal axis of the material. Now let's rewrite the initial result of previous question in the 123 coordinate system:

$$\begin{aligned}
\left[\hat{k} \cdot [\mathcal{A}_{123}] \vec{D} \right] \hat{k} - \underbrace{\left[\hat{k} \cdot \hat{k} \right]}_{=1} [\mathcal{A}_{123}] \vec{D} &= -\frac{1}{n^2} \vec{D} \Longrightarrow \\
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ 0 \end{bmatrix} - \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ 0 \end{bmatrix} &= -\frac{1}{n^2} \begin{bmatrix} D_1 \\ D_2 \\ 0 \end{bmatrix} \Longrightarrow \\
\underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}}_{[\mathcal{A}_t]} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \frac{1}{n^2} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \qquad (6)
\end{aligned}$$

where the elements of the inverse impermeability matrix $[\mathcal{A}_{123}]$ can be found from the following coordinate transformation:

$$[\mathcal{A}_{123}] = [T][\mathcal{A}][T]^T, \qquad \text{with} \qquad (7)$$

$$[T] = \begin{bmatrix} -\frac{a_x a_z}{\sqrt{1-a_z^2}} & -\frac{a_y a_z}{\sqrt{1-a_z^2}} & \sqrt{1-a_z^2} \\ \frac{a_y}{\sqrt{1-a_z^2}} & -\frac{a_x}{\sqrt{1-a_z^2}} & 0 \\ a_x & a_y & a_z \end{bmatrix}.$$

The elements of the $[\mathcal{A}_t]$ matrix, of Eq. (6), can be easily determined and are given by:

$$\begin{aligned}
\alpha_{11} &= \frac{1-a_z^2}{n_{zz}^2} + \frac{a_x^2 a_z^2}{n_{xx}^2 \sqrt{1-a_z^2}} + \frac{a_y^2 a_z^2}{n_{yy}^2 \sqrt{1-a_z^2}}, \\
\alpha_{12} = \alpha_{21} &= \frac{a_x a_y a_z}{1-a_z^2} \left[\frac{1}{n_{yy}^2} - \frac{1}{n_{xx}^2} \right], \\
\alpha_{22} &= \frac{a_x^2}{n_{yy}^2 (1-a_z^2)} + \frac{a_y^2}{n_{xx}^2 (1-a_z^2)}.
\end{aligned}$$

Now the eigenvalues and the eigenvectors can be easily determined analytically and are the following:

$$\rho_f = \frac{1}{n_f^2} = \frac{1}{2}\left[\alpha_{11} + \alpha_{22} + \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21}}\right], \tag{8}$$

$$\vec{D}_f = \frac{\left[1,\ \dfrac{\rho_f - \alpha_{11}}{\alpha_{12}},\ 0\right]^T}{\left\|\left[1,\ \dfrac{\rho_f - \alpha_{11}}{\alpha_{12}},\ 0\right]\right\|}, \tag{9}$$

$$\rho_s = \frac{1}{n_s^2} = \frac{1}{2}\left[\alpha_{11} + \alpha_{22} - \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21}}\right], \tag{10}$$

$$\vec{D}_s = \frac{\left[1,\ \dfrac{\rho_s - \alpha_{11}}{\alpha_{12}},\ 0\right]^T}{\left\|\left[1,\ \dfrac{\rho_s - \alpha_{11}}{\alpha_{12}},\ 0\right]\right\|}. \tag{11}$$

The two eigenvectors can be transformed into the xyz coordinate system by the coordinate transformation $\vec{D}_w^{xyz} = [T]^T \vec{D}_w$ (with $w = f, s$). Applying the last expressions for the numerical application the results of Eqs. (3) and (4) can be obtained.

The orthogonality between the two eigen vectors $\vec{D}_s$ and $\vec{D}_f$ can be easily shown as follows:

$$\begin{aligned}
\vec{D}_f \cdot \vec{D}_s &= K^2 \left[1 + \frac{(\rho_f - \alpha_{11})(\rho_s - \alpha_{11})}{\alpha_{12}^2}\right] = \\
&= K^2 \frac{\alpha_{12}^2 + \alpha_{11}^2 - \alpha_{11}(\rho_f + \rho_s) + \rho_f\rho_s}{\alpha_{12}^2} = 0, \qquad \text{since} \\
\rho_f + \rho_s &= \alpha_{11} + \alpha_{22}, \qquad \text{and} \quad \rho_f\rho_s = \alpha_{11}\alpha_{22} - \alpha_{12}^2.
\end{aligned}$$

In the above expressions K^2 is the product of the normalization constants of the two eigenvectors.

(c) The index ellipsoid can be expressed in general in the quadratic form

$$[x,\ y,\ z][\mathcal{A}][x,\ y,\ z]^T = 1,$$

where $[\mathcal{A}]$ is the impermeability matrix. Transforming the above equation in the 123 coordinate system (as it was introduced in (b)) the index ellipsoid can be written as

$$[x_1,\ x_2,\ x_3][\mathcal{A}_{123}][x_1,\ x_2,\ x_3]^T = 1, \tag{12}$$

where $[\mathcal{A}_{123}]$ is the impermeability matrix given in Eq. (7). According to the index-ellipsoid approach in order to find the two eigen-polarizations and their refractive indices it is necessary to find the intersection of the index ellipsoid with a plane normal to the wavevector direction through the origin. In the 123 coordinate system this plane is the $x_3 = 0$ plane, and the resulting intersection with the index ellipsoid [Eq. (12)] is

$$[x_1,\ x_2][\mathcal{A}_t][x_1,\ x_2]^T = 1, \tag{13}$$

where the last equation specifies the intersection ellipse of the $x_3 = 0$ plane with the index ellipsoid. The 2×2 matrix $[\mathcal{A}_t]$ was specified in (b). From linear algebra it is well known that the eigenvalues of $[\mathcal{A}_t]$ correspond to the two semi-axis of the ellipse which are the eigenvalue refractive indices for propagation along the $\hat{k}$ direction. The directions of the two semi-axes (major and minor) correspond to the two eigen-polarizations (for slow and fast wave). Consequently, the proposed procedure for determining the eigen-polarizations and their refractive indices from index ellipsoid is justified. The eigenvalues and eigenvectors of $[\mathcal{A}_t]$ have been specified in (b) [see Eqs. (8)–(11)].

It might be beneficial to come to the same result from a different initial point. Let's write the index ellipsoid equation in the xyz principal axis coordinate system along with the plane which is normal to the wavevector direction of interest (and passing through the origin):

$$\frac{x^2}{n_{xx}^2} + \frac{y^2}{n_{yy}^2} + \frac{z^2}{n_{zz}^2} = 1, \tag{14}$$

$$\vec{r} \cdot \hat{k} = a_x x + a_y y + a_z z = 0. \tag{15}$$

In order to find the intersection both equations must be satisfied. The two semi-axes of the intersection ellipse correspond to the two eigen-polarizations and their directions correspond to the two eigen-polarizations. To find the semi-axes it is necessary to find the minimum and the maximum of $x^2+y^2+z^2$ on the intersection ellipse. Let's use the method of Lagrange multipliers:

$$F(x,y,z) = (x^2+y^2+z^2) + \lambda_1 \left[\frac{x^2}{n_{xx}^2} + \frac{y^2}{n_{yy}^2} + \frac{z^2}{n_{zz}^2} - 1\right] + \lambda_2 \left[a_x x + a_y y + a_z z\right] \Longrightarrow \tag{16}$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + 2\lambda_1 \frac{x}{n_{xx}^2} + \lambda_2 a_x = 0, \tag{17}$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + 2\lambda_1 \frac{y}{n_{yy}^2} + \lambda_2 a_y = 0, \tag{18}$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + 2\lambda_1 \frac{z}{n_{zz}^2} + \lambda_2 a_z = 0, \tag{19}$$

$$\frac{\partial F}{\partial \lambda_1} = 0 \Longrightarrow \frac{x^2}{n_{xx}^2} + \frac{y^2}{n_{yy}^2} + \frac{z^2}{n_{zz}^2} - 1 = 0, \tag{20}$$

$$\frac{\partial F}{\partial \lambda_2} = 0 \Longrightarrow a_x x + a_y y + a_z z = 0 \tag{21}$$

At this point some manipulations of the above equations are necessary. First, let's multiply Eqs. (17), (18), and (19) by x, y, and z respectively, and add them together. The result [using Eqs. (20) and (21)] is:

$$\lambda_1 = -(x^2+y^2+z^2) = -n^2. \tag{22}$$

Now multiply Eqs. (17), (18), and (19) by a_x, a_y, and a_z respectively, and add them together [making use of the result for λ_1 from Eq. (22)]. Then, the λ_2 becomes

$$\lambda_2 = 2n^2 \left[\frac{a_x x}{n_{xx}^2} + \frac{a_y y}{n_{yy}^2} + \frac{a_z z}{n_{zz}^2}\right]. \tag{23}$$

Next, let's use the solutions for λ_1 and λ_2 to manipulate Eq. (17):

$$
\begin{aligned}
2x\left(1+\frac{\lambda_1}{n_{xx}^2}\right) &= -\lambda_2 a_x \Longrightarrow \\
2x\left(1-\frac{n^2}{n_{xx}^2}\right) &= -2n^2\left[\frac{a_x x}{n_{xx}^2}+\frac{a_y y}{n_{yy}^2}+\frac{a_z z}{n_{zz}^2}\right] a_x \Longrightarrow \\
\frac{x}{n^2}-\frac{x}{n_{xx}^2}+\frac{a_x^2 x}{n_{xx}^2} &= -a_x a_y \frac{y}{n_{yy}^2} - -a_x a_z \frac{z}{n_{zz}^2} \Longrightarrow \\
\left(\frac{n_{xx}^2}{n^2}-a_y^2-a_z^2\right)\frac{x}{n_{xx}^2}+a_x a_y \frac{y}{n_{yy}^2}+a_x a_z \frac{z}{n_{zz}^2} &= 0 \Longrightarrow \\
\frac{a_y^2+a_z^2}{n_{xx}^2}x-\frac{a_x a_y}{n_{yy}^2}y-\frac{a_x a_z}{n_{zz}^2}z &= \frac{1}{n^2}x \Longrightarrow \\
\frac{1-a_x^2}{n_{xx}^2}x-\frac{a_x a_y}{n_{yy}^2}y-\frac{a_x a_z}{n_{zz}^2}z &= \frac{1}{n^2}x
\end{aligned}
\tag{24}
$$

In the last derivation the identity (for the directional cosines) $a_x^2+a_y^2+a_z^2=1$ was used. Performing the same steps for Eqs. (18) and (19) the following set of equations can be written:

$$
\begin{bmatrix}
\frac{1-a_x^2}{n_{xx}^2} & -\frac{a_x a_y}{n_{yy}^2} & -\frac{a_x a_z}{n_{zz}^2} \\
-\frac{a_y a_x}{n_{xx}^2} & \frac{1-a_y^2}{n_{yy}^2} & -\frac{a_y a_z}{n_{zz}^2} \\
-\frac{a_z a_x}{n_{xx}^2} & -\frac{a_z a_y}{n_{yy}^2} & \frac{1-a_z^2}{n_{zz}^2}
\end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{n^2}\begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{25}
$$

Of course the last equation is identical to Eq. (2) that was derived directly from Maxwell's equations. I.e., the geometrical solution of the index ellipsoid is identical to the Maxwell's equations solution. Since the matrix is singular only two eigenvalues are non-zero that correspond to the fast and slow refractive indices. The eigenvectors corresponding to the two eigenvalues give the directions of the two eigen-polarizations.

Of course, one could just solve numerically the system of Eqs. (17)–(21) to find the minimum and maximum of $n^2 = x^2+y^2+z^2$ which correspond to the two eigen-indices (fast and slow). Then, the vector $[x,\ y,\ z]$ would correspond to the eigen-polarization. However, sometimes the numerical solution of a system of nonlinear equations may not provide all the possible solutions depending on the initial guesses.

Another approach is to use other methods for constrained optimization. Such methods are provided by Matlab's function "*fmincon*" which uses the *interior-point* method. To illustrate the procedure let's specify the following minimization problem:

$$
\begin{aligned}
F(x,y,z) &= \underbrace{\min}_{x,y,z}\left\{x^2+y^2+z^2\right\}, \qquad \text{subject to} \\
\frac{x^2}{n_{xx}^2}+\frac{y^2}{n_{yy}^2}+\frac{z^2}{n_{zz}^2} &= 1, \quad \text{and} \\
a_x x + a_y y + a_z z &= 0.
\end{aligned}
$$

Using the above method the point (x_f, y_f, z_f) is found that minimize $F(x, y, z)$. Then the fast refractive index and its corresponding eigen-polarization are $n_f = (x_f^2 + y_f^2 + z_f^2)^{1/2}$ and $\hat{D}_f = [x_f,\ y_f,\ z_f]^T/n_f$. The slow refractive index can be found by minimizing the function $G(x, y, z) = 1/(x^2 + y^2 + z^2)$. The corresponding resulting optimization point is (x_s, y_s, z_s). Then the slow refrative index and its corresponding eigen-polarization are $n_s = (x_s^2 + y_s^2 + z_s^2)^{1/2}$ and $\hat{D}_s = [x_s,\ y_s,\ z_s]^T/n_s$.