# *Gaussian Beams & Resonators*

## Electro-Optics & Applications

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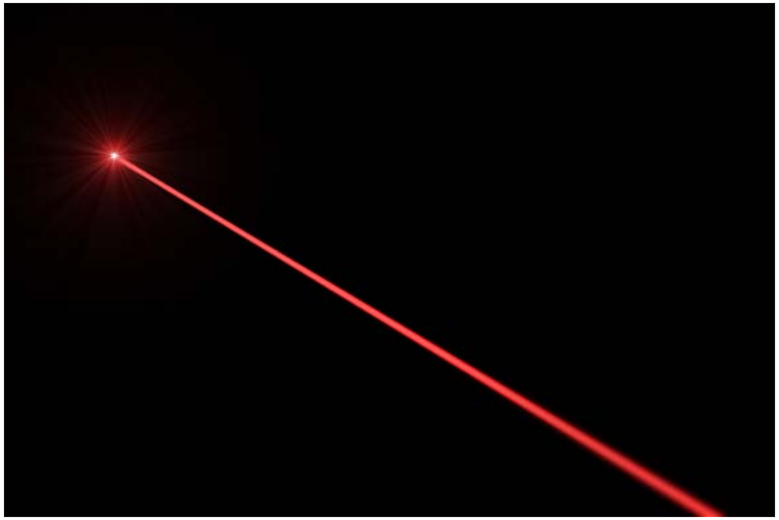
# Gaussian Beams

He-Ne Laser



https://wtamu.edu/~cbaird/sq/images/laser_red.png

Green Laser Pointer

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Necessity for an expression of an electromagnetic field with finite cross-sectional area → Gaussian Beams

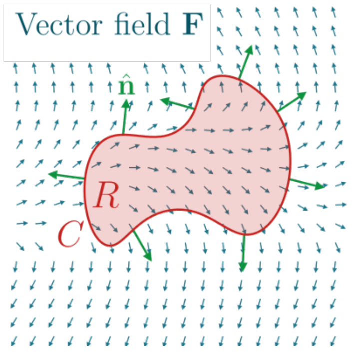
# Gaussian Beams – TEM Solutions

Gauss' Law:

$$\vec{\nabla}\cdot\vec{D}=0 \Longrightarrow \vec{\nabla}\cdot\vec{E}=0,$$

$$\vec{\nabla}\cdot\vec{E}=0 \Longrightarrow \vec{\nabla}_t\cdot\vec{E}_t+\frac{\partial E_z}{\partial z}=0, \quad \vec{E} \;=\; \vec{E}_t \,+\, E_z\hat{z},$$

2D Divergence




https://cdn.kastatic.org/ka-perseus-images/4054da358c8ceddef75ccf33cd9c6ae47367b003.svg

Approximations:

$$\frac{\partial E_z}{\partial z} \;\simeq\; -j\frac{2\pi}{\lambda_0}nE_z,$$

$$\vec{\nabla}_t\cdot\vec{E}_t \;=\; \lim_{\Delta S\to 0}\left\{\frac{1}{\Delta S}\oint_{\Delta\ell}\vec{E}_t\cdot\hat{\imath}_{\perp\ell}d\ell\right\}\simeq\lim_{D\to 0}\left\{\frac{4}{\pi D^2}|\vec{E}_t|\pi D\right\}\simeq\frac{|\vec{E}_t|}{D},$$

TEM Approximation:

$$|E_z|\simeq\frac{\lambda_0}{2\pi nD}|\vec{E}_t|$$

$$\boxed{\frac{|E_z|}{|\vec{E}_t|}\simeq\frac{\lambda_0}{2\pi nD}<10^{-3}}$$

# Gaussian Beams – TEM Solutions

## TEM Approximation:

$$\vec{E}(x,y,z) = \hat{t}E_0\psi(x,y,z)\exp(-jkz)$$

## Helmholtz Scalar Equation

$$\nabla^2 E + k_0^2 n^2 E = 0 \qquad \nabla_t^2\psi - j2k\frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} = 0$$

## Paraxial Wave Equation

$$\nabla_t^2\psi - j2k\frac{\partial\psi}{\partial z} \simeq 0, \qquad \text{if} \quad \left|\frac{\partial^2\psi}{\partial z^2}\right| \ll 2k\left|\frac{\partial\psi}{\partial z}\right|$$

## Fundamental TEM Solution

$$\frac{1}{r_T}\frac{\partial}{\partial r_T}\left(r_T\frac{\partial\psi}{\partial r_T}\right) - j2k\frac{\partial\psi}{\partial z} = 0. \qquad \psi = \exp\left\{-j\left[P(z) + \frac{kr_T^2}{2q(z)}\right]\right\}$$

# Gaussian Beams – TEM Solutions

## Fundamental TEM Solution

$$\left\{ \left[\frac{k^2}{q^2(z)}\left(q'(z)-1\right)\right] r_T^2 \;-\; 2k\left[P'(z)+\frac{j}{q(z)}\right] r_T^0 \right\}\psi = 0, \qquad \forall r_T$$

$$\frac{dq}{dz} = q'(z) \;=\; 1,$$

$$\frac{dP}{dz} = P'(z) \;=\; -\frac{j}{q(z)}.$$

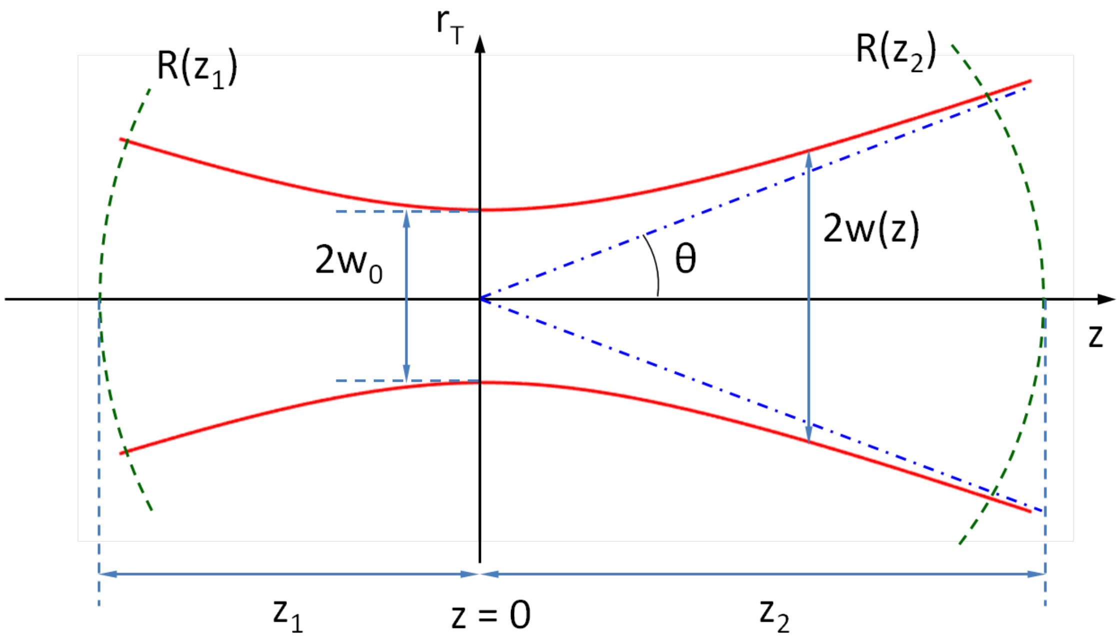
$$\boxed{q(z) = z + j z_0} \qquad |\psi(r_T, z=0)|^2 = \exp(-k r_T^2/z_0)$$

## *Minimum Beam Waist* and *Rayleigh Distance*

$$\boxed{w_0^2 = \frac{2z_0}{k} = \frac{\lambda_0 z_0}{n\pi} \Longrightarrow z_0 = \frac{\pi n w_0^2}{\lambda_0}} \qquad \psi = \exp\left\{-j\left[P(z) + \frac{k r_T^2}{2q(z)}\right]\right\}$$

$$\boxed{jP(z) = \ln\left[1 - j\left(\frac{z}{z_0}\right)\right] = \ln\left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2} - j\tan^{-1}\left(\frac{z}{z_0}\right)}$$

# Gaussian Beams



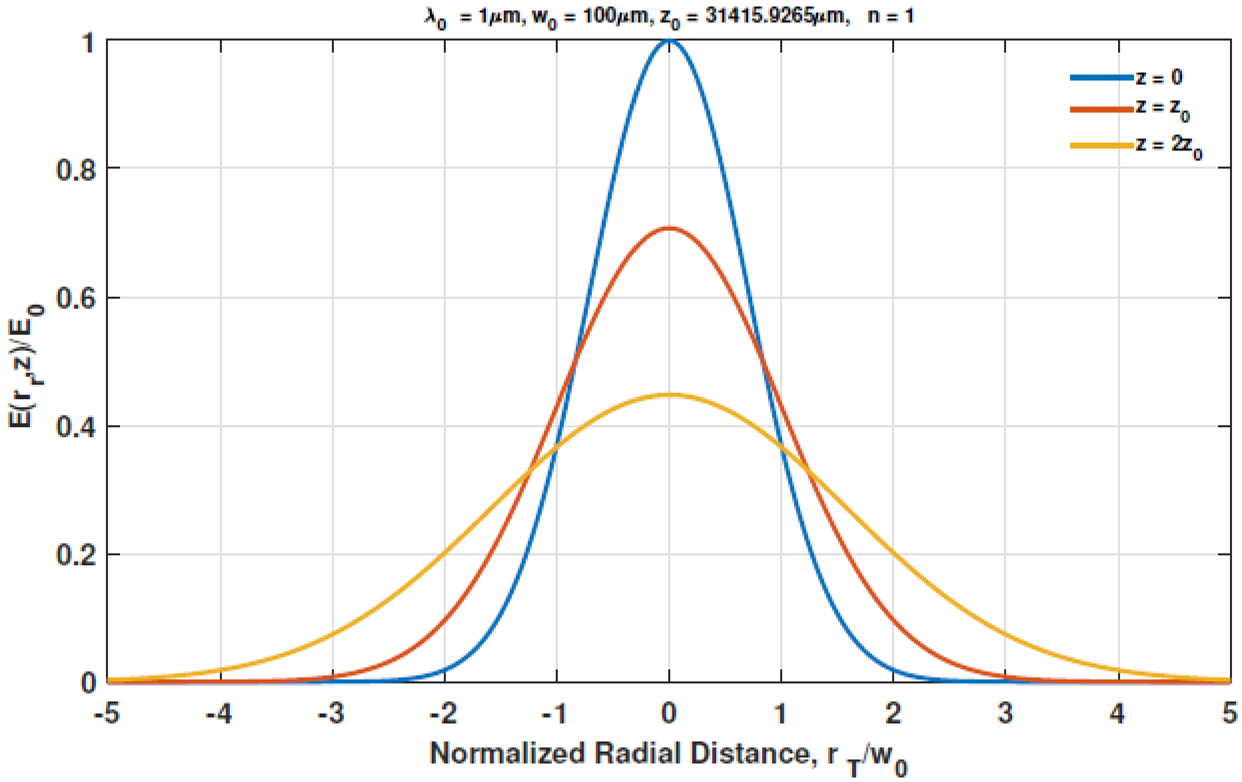

$$E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}$$

$$z_0 = \frac{\pi n w_0^2}{\lambda_0}$$

$$\begin{aligned} \frac{1}{q(z)} &= \frac{z}{z^2 + z_0^2} - j\frac{z_0}{z^2 + z_0^2} = \frac{1}{R(z)} - j\frac{\lambda_0}{\pi n w^2(z)}, \\ w^2(z) &= w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right], \\ R(z) &= z\left[1 + \left(\frac{z_0}{z}\right)^2\right]. \end{aligned}$$

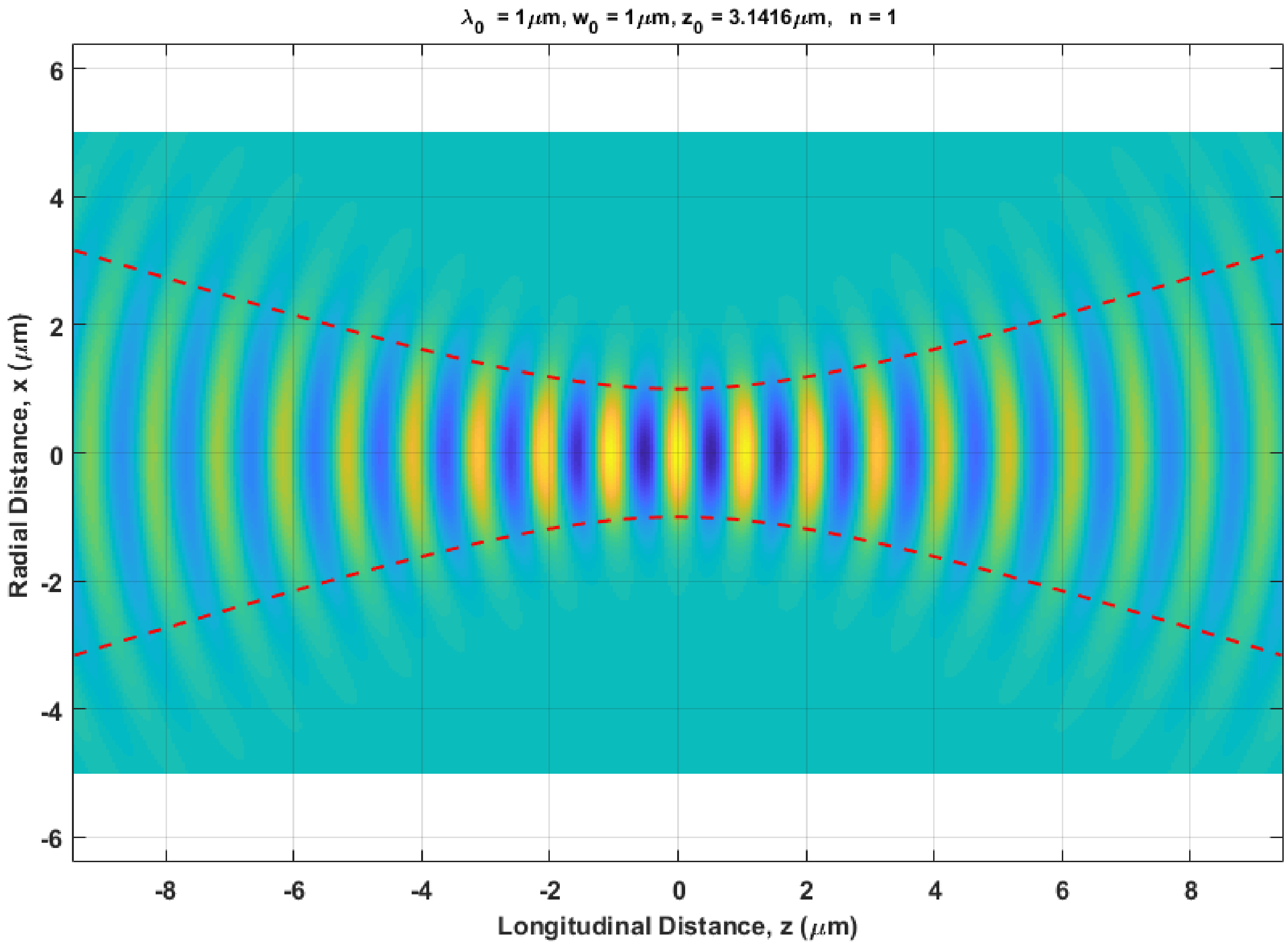
# Gaussian Beams

## Radial Profiles at various z Distances from Focal Plane



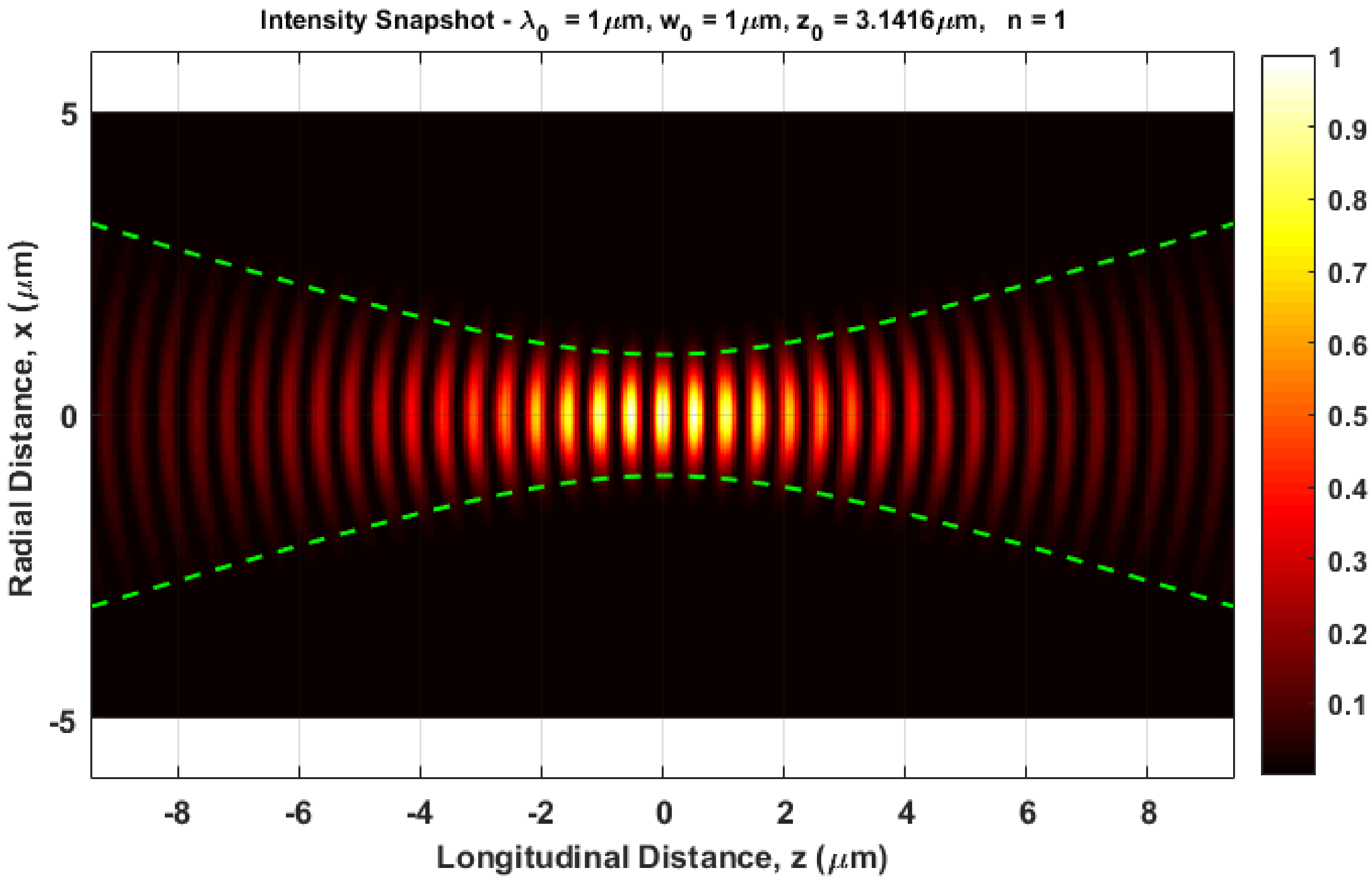
# Gaussian Beams

## Electric Field Snapshot in Time



# Gaussian Beams

## Intensity Snapshot in Time



# Gaussian Beams

## Gaussian Beam Divergence

$$w(z) \simeq \frac{w_0 z}{z_0} = \frac{\lambda_0 z}{\pi n w_0},$$

$$\tan\theta = \frac{dw(z)}{dz} = \frac{\lambda_0}{\pi n w_0} \Longrightarrow \boxed{\Delta\theta = 2\theta \simeq \frac{2\lambda_0}{\pi n w_0} = \frac{4}{\pi}\,\frac{\lambda_0/n}{D_0}}$$

## Gaussian Beam Power

$$P(z) = \frac{1}{2\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |E_{00}|^2 dx dy = \frac{1}{2\eta}\int_{-\infty}^{+\infty}\int_{0}^{2\pi} |E_{00}(r_T, z)|^2 r_T dr_T d\phi = \frac{1}{2\eta}|\mathcal{E}_0|^2 \left[\frac{\pi w_0^2}{2}\right]$$

# Gaussian Beams

## Gaussian Beam Radius of Curvature

### Spherical Wave

$$
\begin{aligned}
E_{sph} &= \frac{1}{r}\exp(-jkr), \quad \text{where} \\
r &= \sqrt{x^2+y^2+z^2} = \sqrt{r_T^2+z^2} = z\sqrt{1+\frac{r_T^2}{z^2}} \\
r &\simeq z + \frac{1}{2}\frac{r_T^2}{z} \simeq z + \frac{1}{2}\frac{r_T^2}{r}, \quad \text{and} \\
E_{sph} &\simeq \frac{1}{r}\exp(-jkz)\exp\left(-j\frac{kr_T^2}{2r}\right).
\end{aligned}
$$

### Fundamental Gaussian Beam

$$
E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)}\exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \boxed{\underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}}
$$

# Gaussian Beams

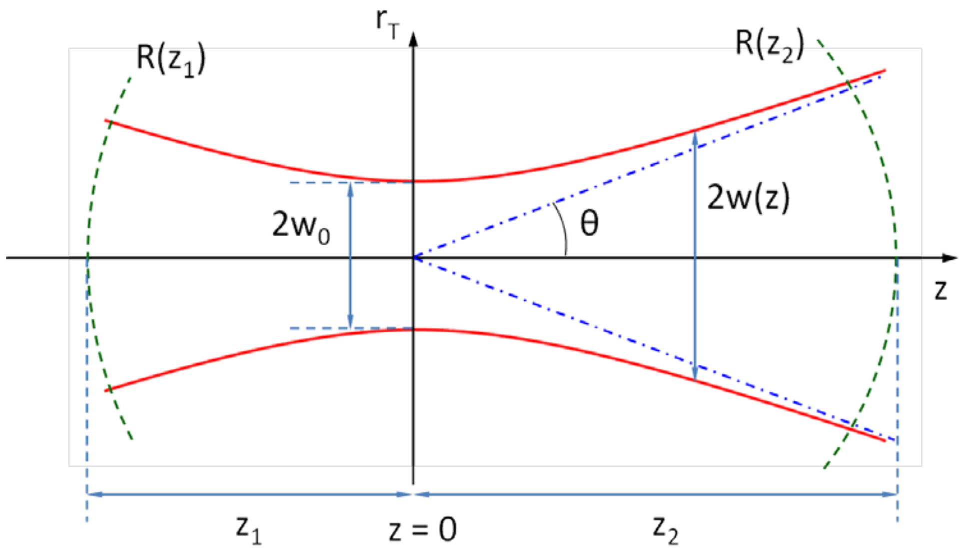
## Fundamental Gaussian Beam

$$E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}$$

## Gaussian Beam Radius of Curvature

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

$$R(z) \simeq \begin{cases} +\infty, & \text{centered at} \quad z_c = \mp\infty, \quad \text{for} \quad |z| \ll z_0, \\ \mp 2z_0, & \text{centered at} \quad z_c = \mp z_0 \quad \text{for} \quad |z| = z_0, \\ z, & \text{centered at} \quad z_c = 0, \quad \text{for} \quad |z| \gg z_0. \end{cases}$$



# Gaussian Beams

## Gaussian Beam Radius of Curvature

$$R(z) \simeq \begin{cases} +\infty, & \text{centered at} \quad z_c = \mp\infty, \quad \text{for} \quad |z| \ll z_0, \\ \mp 2z_0, & \text{centered at} \quad z_c = \mp z_0 \quad \text{for} \quad |z| = z_0, \\ z, & \text{centered at} \quad z_c = 0, \quad \text{for} \quad |z| \gg z_0. \end{cases}$$

## Gaussian Beam Longitudinal Phase Shift – *Gouy's* Shift

$$E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}$$

$$\Phi(z) = kz - \tan^{-1}(z/z_0)$$

## Phase Velocity (exact)

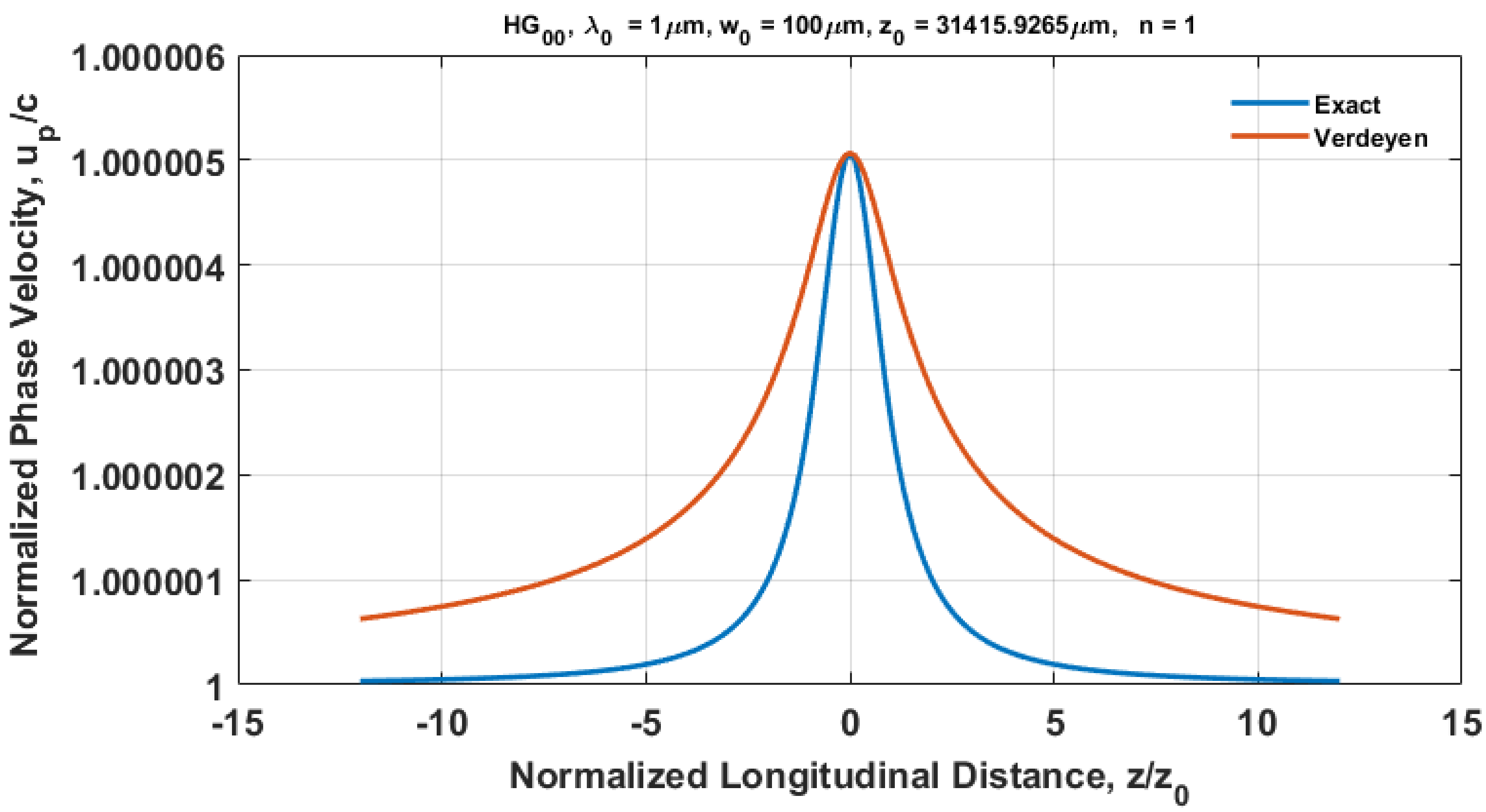
$$u_p = \frac{\omega}{d\Phi/dz} = \frac{\dfrac{c}{n}}{1 - \dfrac{\lambda_0 z_0}{2\pi n} \dfrac{1}{z^2 + z_0^2}}$$

## Phase Velocity (approximate)

$$u_p = \frac{\omega}{\Phi/z} = \frac{\dfrac{c}{n}}{1 - \dfrac{\lambda_0}{2\pi n z} \tan^{-1}\left(\dfrac{z}{z_0}\right)}.$$

# Gaussian Beams

## Gaussian Beam Longitudinal Phase Shift – *Gouy's* Shift



# Hermite-Gaussian Beams

$$E_{mp}(x,y,z) = E_{mp}^{HG} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_p\left(\frac{\sqrt{2}y}{w(z)}\right) \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)}\right] \exp\left\{-j\left[kz-(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\} \exp\left[-j\frac{k(x^2+y^2)}{2R(z)}\right]$$

where,

$$w^2(z) = w_0^2\left(1+\frac{z^2}{z_0^2}\right),$$

$$R(z) = z\left(1+\frac{z_0^2}{z^2}\right),$$

$$H_m(u) = (-1)^m e^{u^2}\frac{d^m}{du^m}\left(e^{-u^2}\right),$$

$$H_{m+1}(u) = 2uH_m(u) - 2mH_{m-1}(u), \quad \text{with} \quad H_0(u)=1 \quad \text{and} \quad H_1(u)=u,$$

## Some *Hermite* Polynomials

$$H_0(x) = 1$$

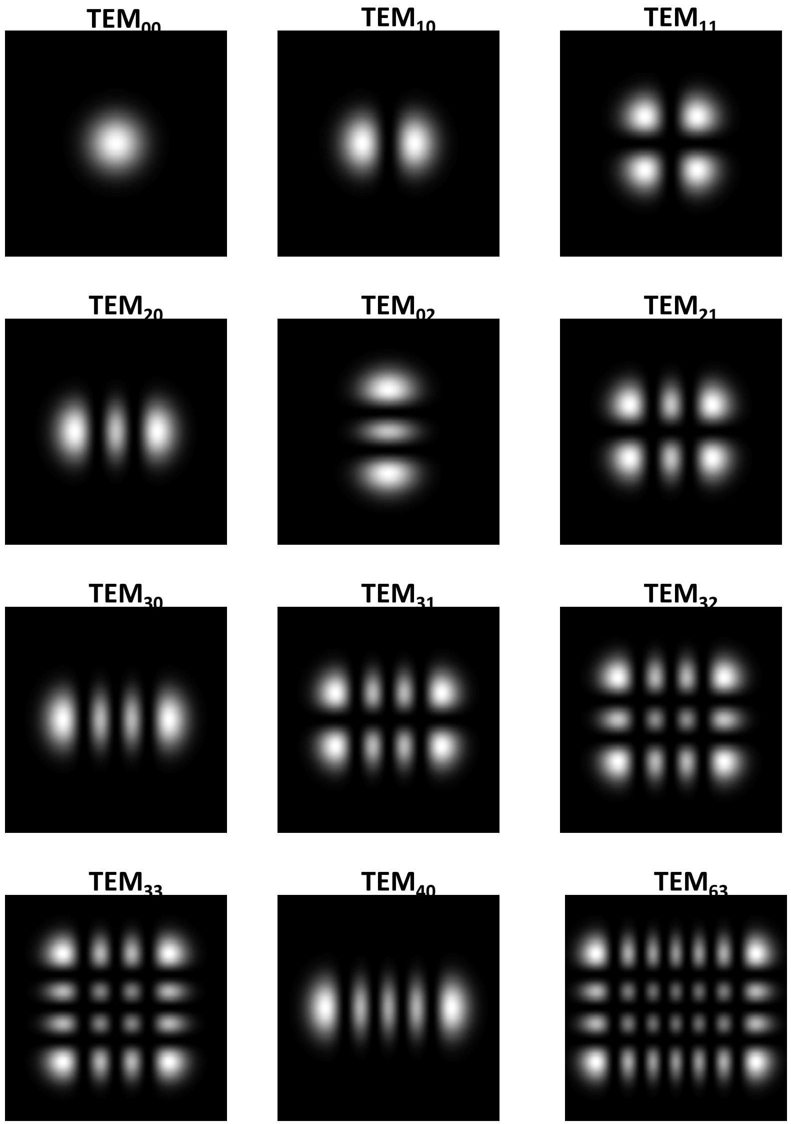
$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

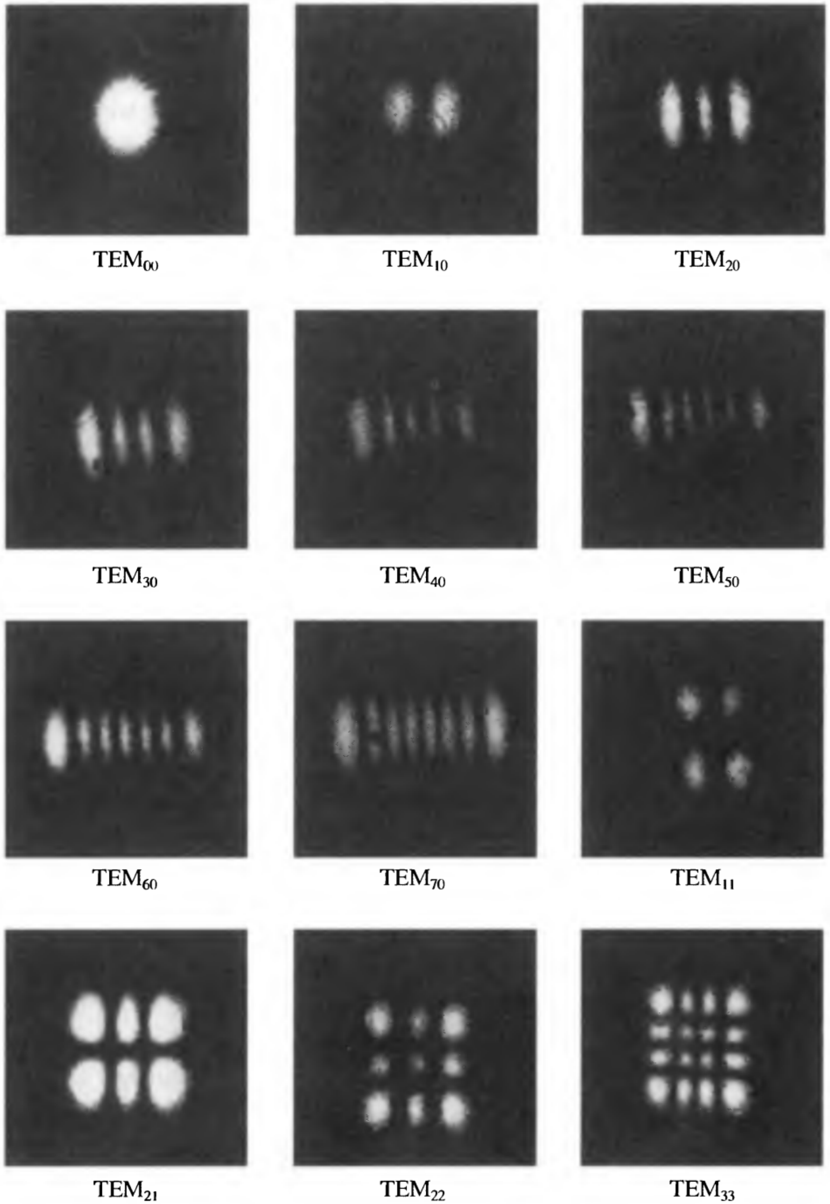
$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

# Hermite-Gaussian Beams Patterns



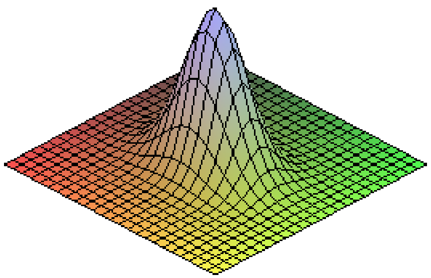
# Experimental Patterns of Gaussian Beams



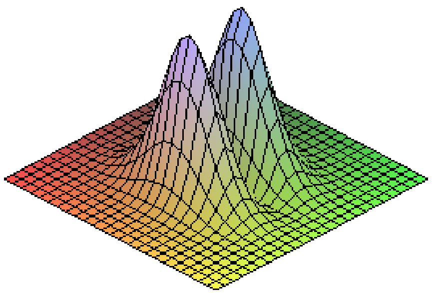

From A. Yariv and P. Yeh, "Photonics" 6th Ed. Oxford University Press, 2007

# Hermite-Gaussian Beams Patterns

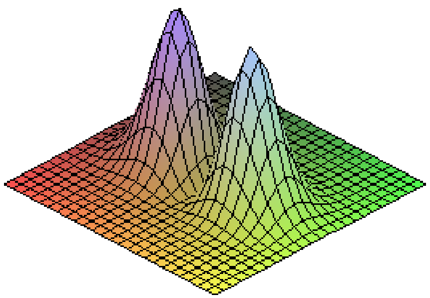
$TEM_{00}$



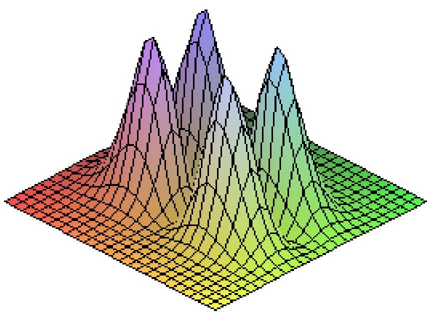
$TEM_{10}$



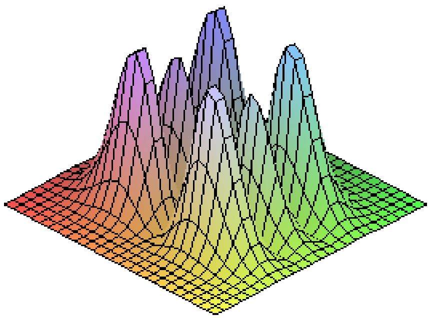
$TEM_{01}$



$TEM_{11}$



$TEM_{21}$



# Laguerre-Gaussian Beams

$$E_{pm}(r_T,\phi,z) = E_{pm}^{LG}\frac{w_0}{w(z)}\left(\frac{\sqrt{2}r_T}{w(z)}\right)^{|m|} L_p^{|m|}\left(\frac{2r_T^2}{w^2(z)}\right)\exp\left[-\frac{r_T^2}{w^2(z)}\right]$$
$$\exp\left\{-j\left[kz-(1+|m|+2p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}\exp(jm\phi)\exp\left[-j\frac{kr_T^2}{2R(z)}\right]$$

where,

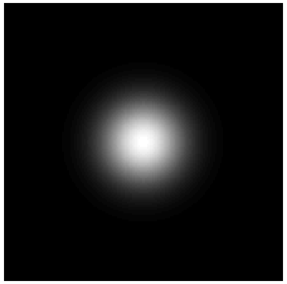
$$w^2(z) = w_0^2\left(1+\frac{z^2}{z_0^2}\right),$$
$$R(z) = z\left(1+\frac{z_0^2}{z^2}\right),$$
$$L_p^m(u) = \frac{e^u}{p!}u^{-m}\frac{d^p}{du^p}\left(e^{-u}u^{p+m}\right),$$
$$(k+1)L_{k+1}^p(u) = (2k+1+p-u)L_k^p(u)-(k+p)L_{k-1}^p(u), \quad \text{with}$$
$$L_0^m(u) = 1 \quad \text{and} \quad L_1^m(u) = -u+(m+1),$$

## Some *Laguerre* Polynomials

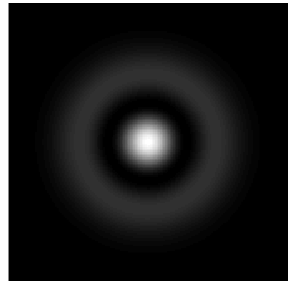
$$L_0^m(x) = 1$$
$$L_1^m(x) = -x+(m+1)$$
$$L_2^m(x) = \tfrac{1}{2}[x^2-2(m+2)x+(m+1)(m+2)]$$
$$L_3^m(x) = \tfrac{1}{6}[-x^3+3(m+3)x^2-3(m+2)(m+3)x+(m+1)(m+2)(m+3)]$$

# Laguerre-Gaussian Beams Patterns

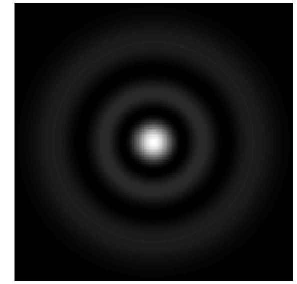
TEM(0,0) - cos(mϕ)



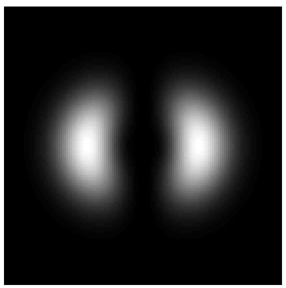
TEM(1,0) - cos(mϕ)



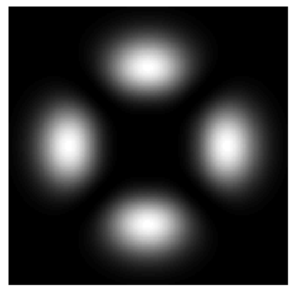
TEM(2,0) - cos(mϕ)



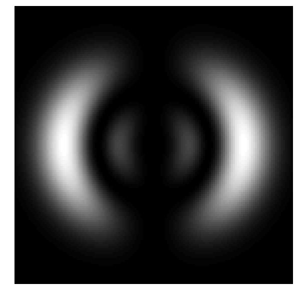
TEM(0,1) - cos(mϕ)



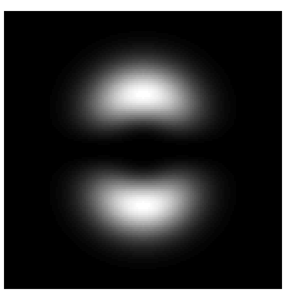
TEM(0,2) - cos(mϕ)



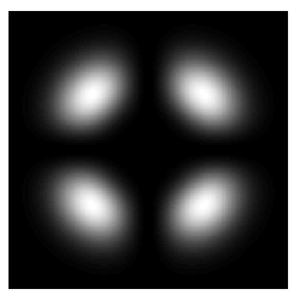
TEM(1,1) - cos(mϕ)



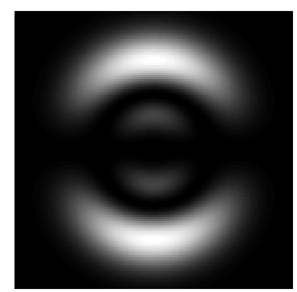
TEM(0,1) - sin(mϕ)



TEM(0,2) - sin(mϕ)



TEM(1,1) - sin(mϕ)



# Validity of Paraxial Approximation

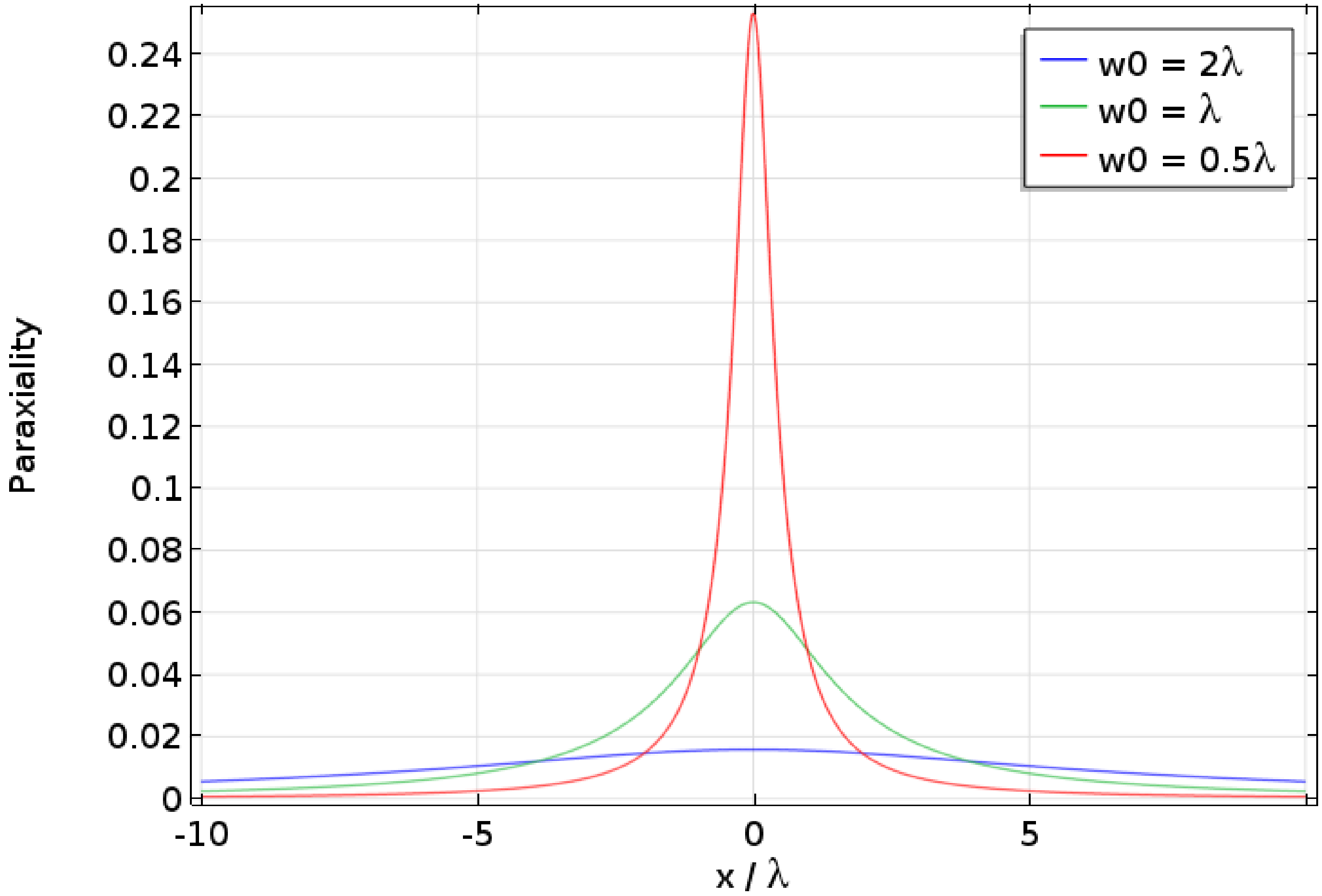
$$\begin{aligned}\mathcal{P} &= 1-\frac{N+1}{k_0^2 n^2 w_0^2}, \qquad \text{where}\\ N &= m+p, \qquad \text{for Hermite-Gaussian beams,}\\ N &= 2p+|m|, \qquad \text{for Laguerre-Gaussian beams.}\end{aligned}$$

$$\mathcal{P} \simeq 1 \qquad \text{(the closer to 1 the better the paraxial approximation)}$$

P. Vaveliuk et al, "Limits of the paraxial approximation in laser beams", Opt. Lett., 32, (927-929), 2007

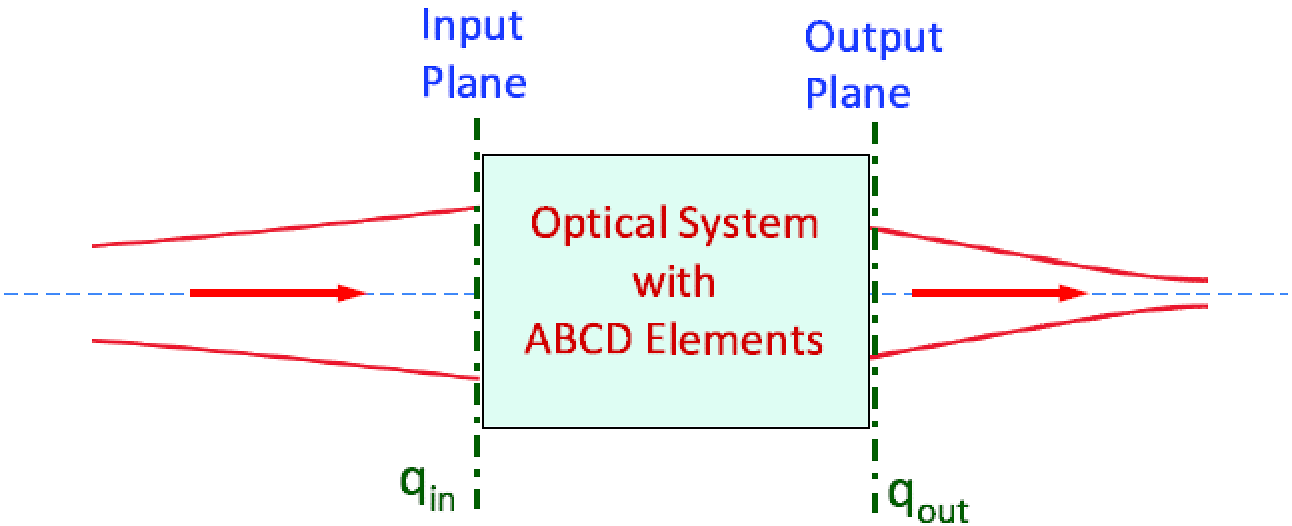
# Validity of Paraxial Approximation

$$\text{Paraxiality} = \left| \frac{\frac{\partial^2 \psi}{\partial z^2}}{2jk\frac{\partial \psi}{\partial z}} \right|$$




https://www.comsol.com/blogs/understanding-the-paraxial-gaussian-beam-formula/

# Gaussian Beams and ABCD Law

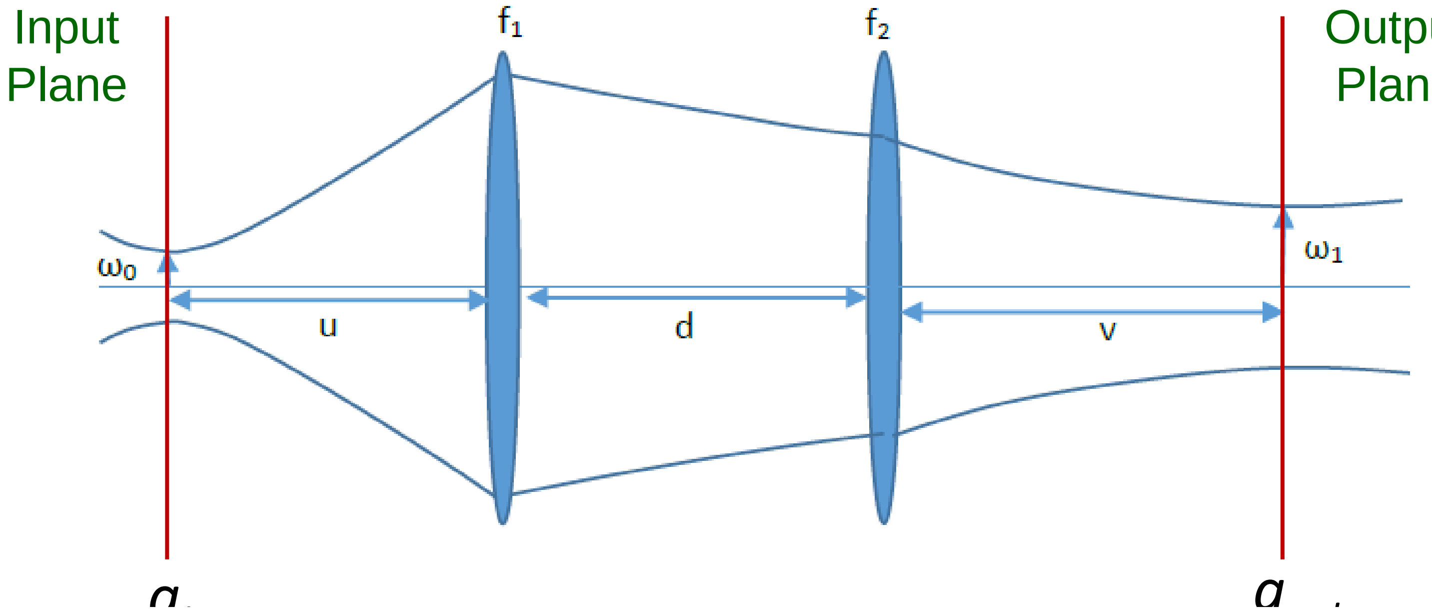


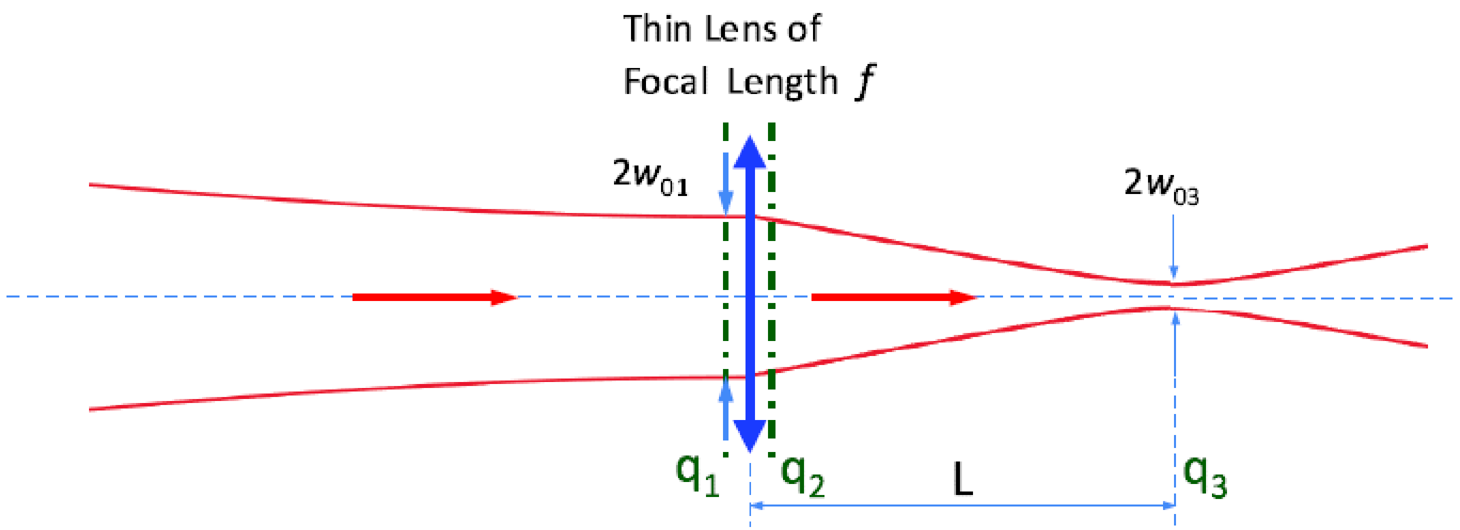

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

$$\frac{1}{q_{out}} = \frac{C + D(1/q_{in})}{A + B(1/q_{in})}$$

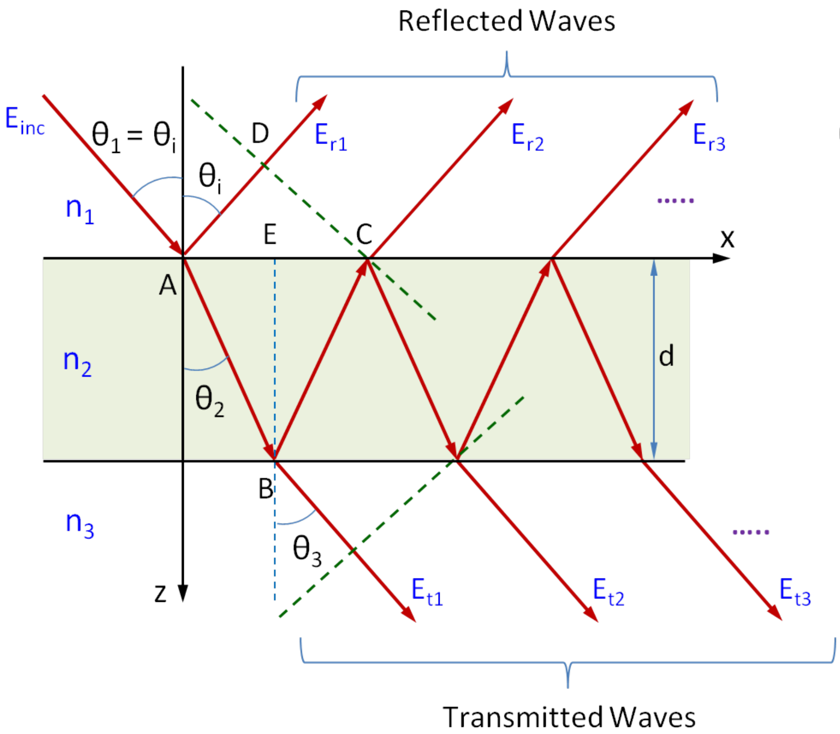
$$\frac{1}{q(z)} = \frac{z}{z^2 + z_0^2} - j\frac{z_0}{z^2 + z_0^2} = \frac{1}{R(z)} - j\frac{\lambda_0}{\pi n w^2(z)}$$

# Gaussian Beams and ABCD Law Examples



# Fabry-Perot Interferometer




$$\begin{aligned}
\delta &= k_0 n_2[(AB)+(BC)] - k_0 n_1(AD), \qquad \text{where,} \\
(AB) = (BC) &= \frac{d}{\cos\theta_2}, \\
(AD) &= (AC)\sin\theta_1 = 2(AE)\sin\theta_1 = 2d\tan\theta_2\sin\theta_1, \\
\delta &= 2k_0 n_2 d\cos\theta_2,
\end{aligned}$$

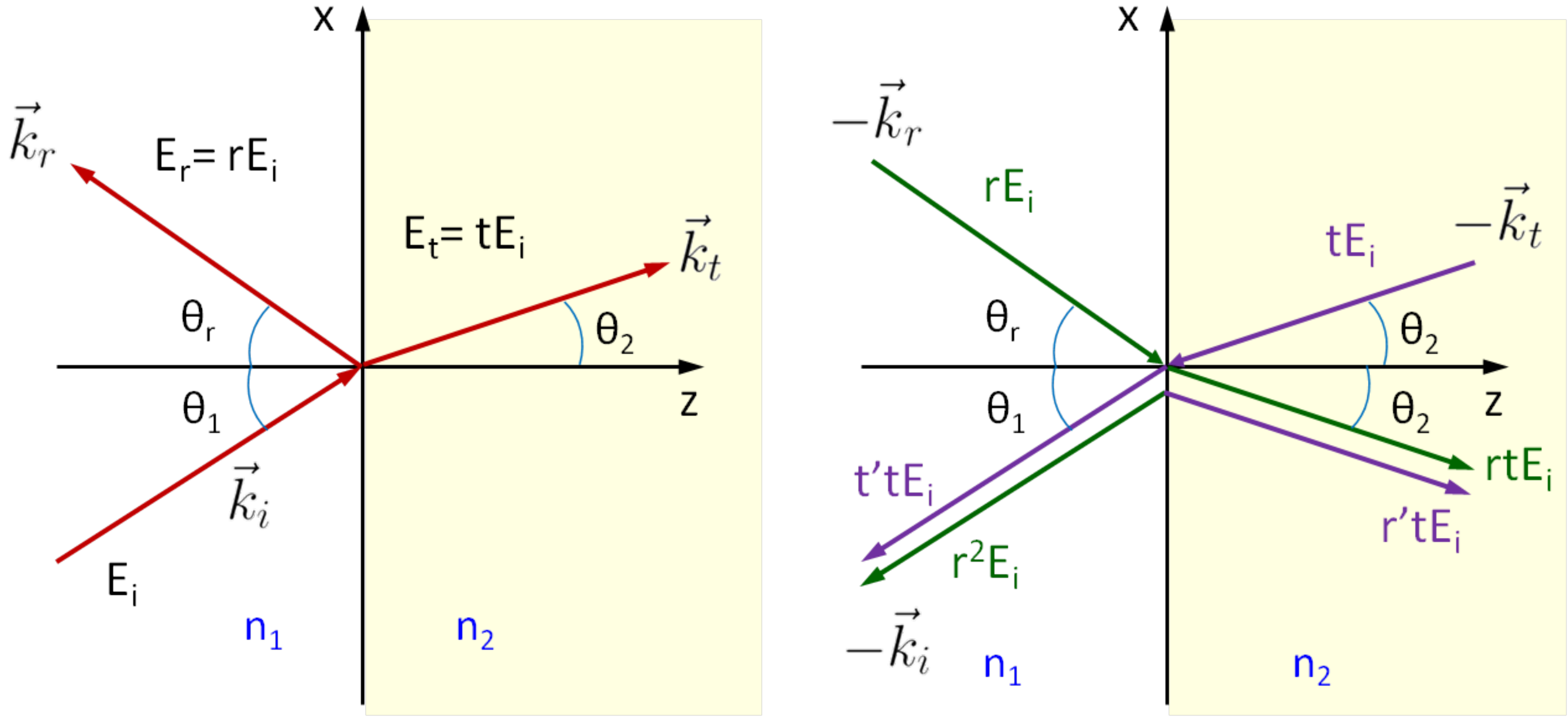
## Reflected Waves

$$\begin{aligned}
E_{r1} &= r_{12}E_i, \\
E_{r2} &= t_{12}t_{21}r_{23}e^{-j\delta}E_i, \\
E_{r3} &= t_{12}t_{21}r_{21}r_{23}^2 e^{-j2\delta}E_i, \\
\vdots &= \vdots, \\
E_{rN} &= t_{12}t_{21}r_{21}^{N-2}r_{23}^{N-1}e^{-j(N-1)\delta}E_i, \\
\vdots &= \vdots.
\end{aligned}$$

$$\begin{aligned}
E_r &= \sum_{m=1}^{\infty} E_{rm} = r_{12}E_i + t_{12}t_{21}r_{23}e^{-j\delta}E_i\left[1 + r_{21}r_{23}e^{-j\delta} + r_{21}^2 r_{23}^2 e^{-j2\delta} + \cdots\right] = \\
&= r_{12}E_i + t_{12}t_{21}r_{23}e^{-j\delta}E_i\frac{1}{1 - r_{21}r_{23}e^{-j\delta}} = r_{12}E_i - (1 - r_{12}^2)E_i\,\frac{r_{32}e^{-j\delta}}{1 - r_{12}r_{32}e^{-j\delta}}
\end{aligned}$$

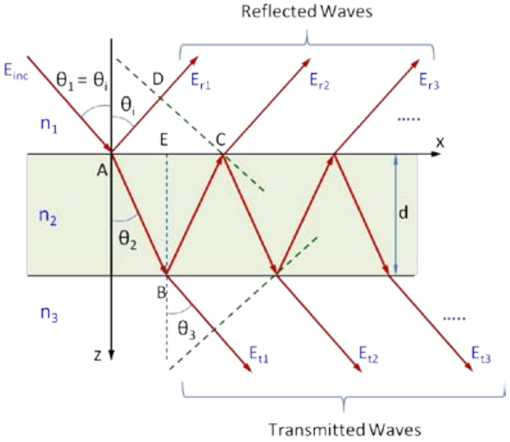
$$\frac{E_r}{E_i} = \frac{r_{12} - r_{32}e^{-j\delta}}{1 - r_{12}r_{32}e^{-j\delta}}.$$

# Stokes Relations




$$
\begin{aligned}
rt + r't &= 0 \Longrightarrow r' = -r, \\
tt' + r^2 &= 1.
\end{aligned}
$$

# Fabry-Perot Interferometer




## Transmitted Waves

$$E_t = \sum_{m=1}^{\infty} E_{tm} = t_{12}t_{23}E_i \left[1 + r_{21}r_{23}e^{-j\delta} + r_{21}^2 r_{23}^2 e^{-j2\delta} + \cdots\right]$$

$$\boxed{\frac{E_t}{E_i} = \frac{t_{12}t_{23}}{1 - r_{12}r_{32}e^{-j\delta}}}$$

$$\frac{I_r}{I_i} = \frac{P_r}{P_i} = \frac{|E_r|^2}{|E_i|^2} = \frac{|r_{12}|^2 + |r_{32}|^2 - 2|r_{12}||r_{32}|\cos\delta}{1 + |r_{12}|^2|r_{32}|^2 - 2|r_{12}||r_{32}|\cos\delta} =$$

$$= \frac{(|r_{12}| - |r_{32}|)^2 + 4|r_{12}||r_{32}|\sin^2(\delta/2)}{(1 - |r_{12}||r_{32}|)^2 + 4|r_{12}||r_{32}|\sin^2(\delta/2)} = \frac{(\sqrt{R_1} - \sqrt{R_2})^2 + 4\sqrt{R_1R_2}\sin^2(\delta/2)}{(1 - \sqrt{R_1R_2})^2 + 4\sqrt{R_1R_2}\sin^2(\delta/2)},$$

$$\frac{I_t}{I_i} = \frac{P_t}{P_i} = \frac{n_3\cos\theta_3}{n_1\cos\theta_1}\frac{|E_t|^2}{|E_i|^2} = \frac{|t_{12}|^2|t_{23}|^2}{1 + |r_{12}|^2|r_{32}|^2 - 2|r_{12}||r_{32}|\cos\delta},$$

using Eqs. (118) or (119)

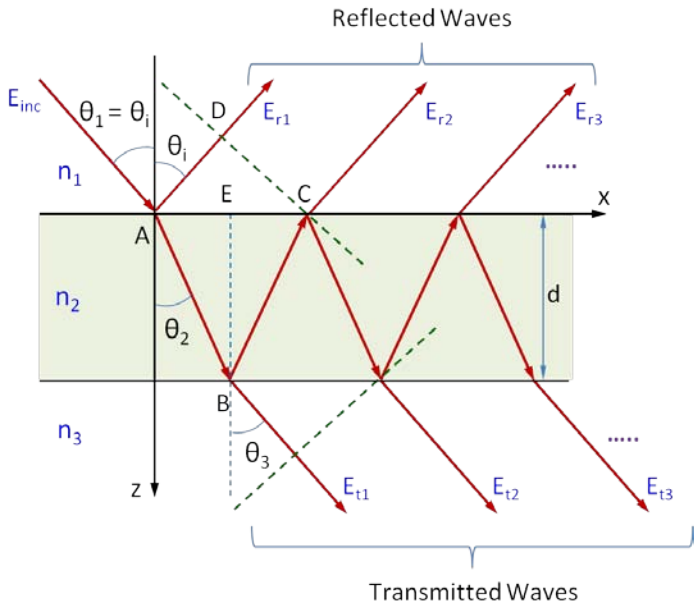
$$|t_{12}|^2 = \frac{n_1\cos\theta_1}{n_2\cos\theta_2}\left(1 - |r_{12}|^2\right),$$

$$|t_{23}|^2 = \frac{n_2\cos\theta_2}{n_3\cos\theta_3}\left(1 - |r_{32}|^2\right),$$ results in,

$$\frac{I_t}{I_i} = \frac{(1 - |r_{12}|^2)(1 - |r_{32}|^2)}{1 + |r_{12}|^2|r_{32}|^2 - 2|r_{12}||r_{32}|\cos\delta} =$$

$$\frac{(1 - |r_{12}|^2)(1 - |r_{32}|^2)}{(1 - |r_{12}||r_{32}|)^2 + 4|r_{12}||r_{32}|\sin^2(\delta/2)} = \frac{(1 - R_1)(1 - R_2)}{(1 - \sqrt{R_1R_2})^2 + 4\sqrt{R_1R_2}\sin^2(\delta/2)}$$

# Fabry-Perot Interferometer




$$R_1 = |r_{12}|^2 \text{ and } R_2 = |r_{32}|^2$$

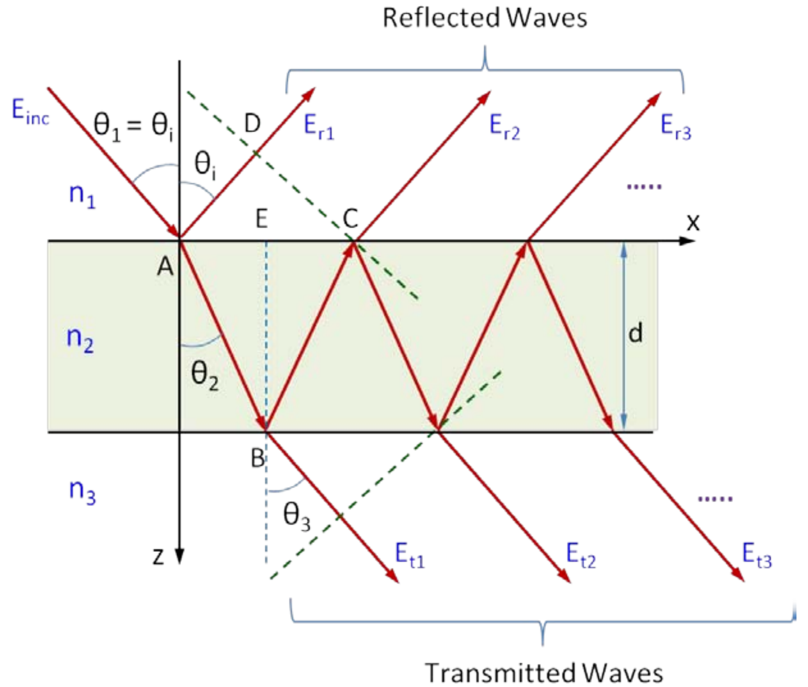
$$\left(\frac{I_t}{I_i}\right)_{max} = \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1R_2})^2}, \qquad \text{for} \quad \delta = 2m\pi,$$

$$\left(\frac{I_t}{I_i}\right)_{min} = \frac{(1-R_1)(1-R_2)}{(1+\sqrt{R_1R_2})^2}, \qquad \text{for} \quad \delta = (2m+1)\pi,$$

$$\left(\frac{I_r}{I_i}\right)_{max} = \frac{(\sqrt{R_1}+\sqrt{R_2})^2}{(1+\sqrt{R_1R_2})^2}, \qquad \text{for} \quad \delta = (2m+1)\pi,$$

$$\left(\frac{I_r}{I_i}\right)_{min} = \frac{(\sqrt{R_1}-\sqrt{R_2})^2}{(1-\sqrt{R_1R_2})^2}, \qquad \text{for} \quad \delta = 2m\pi.$$

# Symmetric Fabry-Perot Interferometer



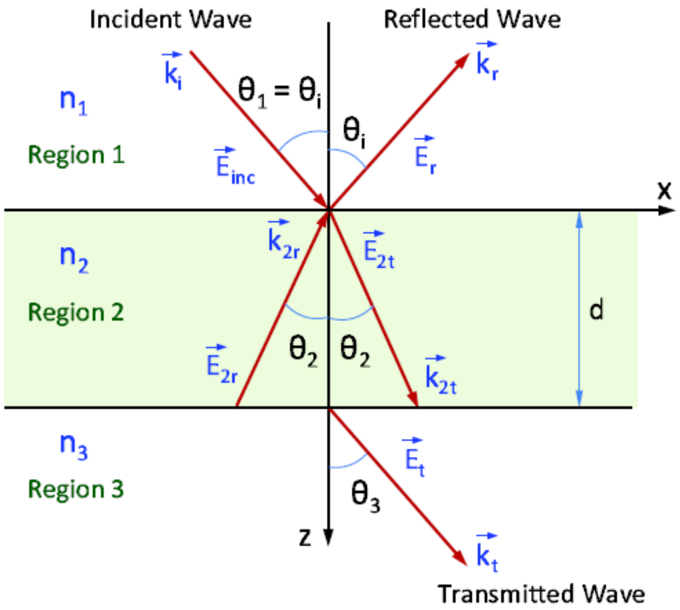

$n_1 = n_3 = n_0$ and $n_2 = n_f$ $\quad r_{12} = r_{32} = r \; (\phi_{12} = \phi_{32} = 0) \quad R_1 = R_2 = R = r^2$

$$\delta = k_0\Delta, \qquad \Delta = 2dn_2 \cos\theta_2, \qquad R = r^2$$

$$\frac{I_t}{I_i} = \frac{(1-r^2)^2}{1+r^4-2r^2\cos\delta} = \frac{(1-R)^2}{(1-R)^2+4R\sin^2(\delta/2)}$$

$$\frac{I_r}{I_i} = \frac{2r^2(1-\cos\delta)}{1+r^4-2r^2\cos\delta} = \frac{4R\sin^2(\delta/2)}{(1-R)^2+4R\sin^2(\delta/2)}$$

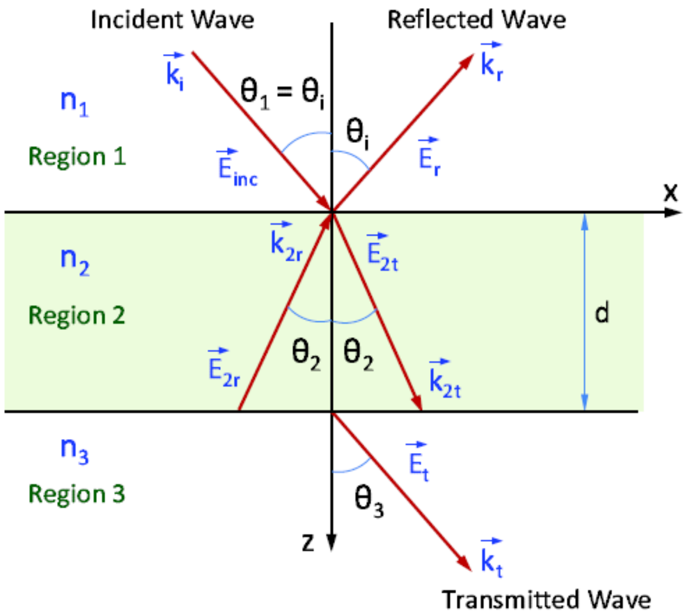
# Electromagnetic Analysis of Fabry-Perot Interferometer




$$
\begin{aligned}
\vec{E}_1 &= \vec{E}_i + \vec{E}_r = \hat{\imath}_y \left[ E_i e^{-j\vec{k}_i\cdot\vec{r}} + E_r e^{-j\vec{k}_r\cdot\vec{r}} \right], \qquad \text{where} \\
\vec{k}_i &= k_0 n_1 (\hat{\imath}_x \sin\theta_1 + \hat{\imath}_z \cos\theta_1), \qquad \vec{k}_r = k_0 n_1 (\hat{\imath}_x \sin\theta_1 - \hat{\imath}_z \cos\theta_1), \\
\vec{E}_2 &= \vec{E}_{2t} + \vec{E}_{2r} = \hat{\imath}_y \left[ E_{2t} e^{-j\vec{k}_{2t}\cdot\vec{r}} + E_{2r} e^{-j\vec{k}_{2r}\cdot\vec{r}} \right], \qquad \text{where} \\
\vec{k}_{2t} &= k_0 n_2 (\hat{\imath}_x \sin\theta_2 + \hat{\imath}_z \cos\theta_2), \qquad \vec{k}_{2r} = k_0 n_2 (\hat{\imath}_x \sin\theta_2 - \hat{\imath}_z \cos\theta_2), \\
\vec{E}_3 &= \vec{E}_t = \hat{\imath}_y E_t e^{-j\vec{k}_t\cdot(\vec{r} - d\hat{\imath}_z)}, \qquad \text{where} \\
\vec{k}_t &= k_0 n_3 (\hat{\imath}_x \sin\theta_3 + \hat{\imath}_z \cos\theta_3).
\end{aligned}
$$

$$
\begin{aligned}
H_{1x} &= \frac{1}{Z_1}\left[ -E_i \cos\theta_1 e^{-jk_0 n_1 (x\sin\theta_1 + z\cos\theta_1)} + E_r \cos\theta_1 e^{-jk_0 n_1 (x\sin\theta_1 - z\cos\theta_1)} \right], \\
H_{2x} &= \frac{1}{Z_2}\left[ -E_{2t} \cos\theta_2 e^{-jk_0 n_2 (x\sin\theta_2 + z\cos\theta_2)} + E_{2r} \cos\theta_2 e^{-jk_0 n_1 (x\sin\theta_2 - z\cos\theta_2)} \right], \\
H_{3x} &= -\frac{1}{Z_3} E_t \cos\theta_3 e^{-jk_0 n_3 (x\sin\theta_3 + (z-d)\cos\theta_3)}.
\end{aligned}
$$

# Electromagnetic Analysis of Fabry-Perot Interferometer




## Οριακές Συνθήκες

$$\beta_2 = k_0 n_2 \cos\theta_2$$

$$\begin{bmatrix} E_{2t} \\ E_{2r} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}\left(1 + \frac{n_1\cos\theta_1}{n_2\cos\theta_2}\right) & \frac{1}{2}\left(1 - \frac{n_1\cos\theta_1}{n_2\cos\theta_2}\right) \\ \frac{1}{2}\left(1 - \frac{n_1\cos\theta_1}{n_2\cos\theta_2}\right) & \frac{1}{2}\left(1 + \frac{n_1\cos\theta_1}{n_2\cos\theta_2}\right) \end{bmatrix}}_{\tilde{M}_{2,1}} \begin{bmatrix} E_i \\ E_r \end{bmatrix},$$

$$\begin{bmatrix} E_t \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}\left(1 + \frac{n_2\cos\theta_2}{n_3\cos\theta_3}\right)e^{-j\beta_2 d} & \frac{1}{2}\left(1 - \frac{n_2\cos\theta_2}{n_3\cos\theta_3}\right)e^{+j\beta_2 d} \\ \frac{1}{2}\left(1 - \frac{n_2\cos\theta_2}{n_3\cos\theta_3}\right)e^{-j\beta_2 d} & \frac{1}{2}\left(1 + \frac{n_2\cos\theta_2}{n_3\cos\theta_3}\right)e^{+j\beta_2 d} \end{bmatrix}}_{\tilde{M}_{3,2}} \begin{bmatrix} E_{2t} \\ E_{2r} \end{bmatrix},$$

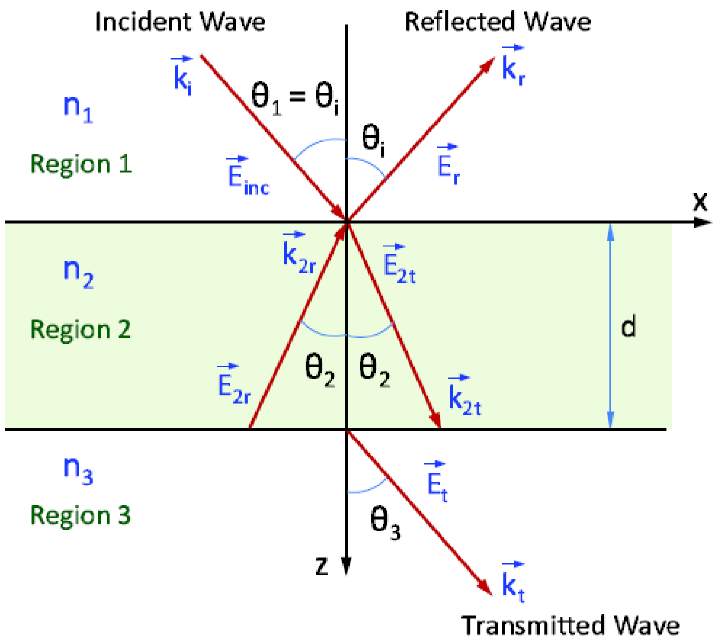
$$\begin{bmatrix} E_t \\ 0 \end{bmatrix} = \tilde{M}_{3,2}\tilde{M}_{2,1}\begin{bmatrix} E_i \\ E_r \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} E_i \\ E_r \end{bmatrix} \Longrightarrow$$

$$\frac{E_r}{E_i} = r = -\frac{m_{21}}{m_{22}},$$

$$\frac{E_t}{E_i} = t = m_{11} - \frac{m_{12}m_{21}}{m_{22}}, \qquad \text{where,}$$

$$\begin{aligned} K_{12} &= \frac{n_1\cos\theta_1}{n_2\cos\theta_2}, \\ K_{23} &= \frac{n_2\cos\theta_2}{n_3\cos\theta_3}, \\ m_{11} &= \frac{1}{4}\left[(1+K_{12})(1+K_{23})e^{-j\beta_2 d} + (1-K_{12})(1-K_{23})e^{+j\beta_2 d}\right], \\ m_{12} &= \frac{1}{4}\left[(1-K_{12})(1+K_{23})e^{-j\beta_2 d} + (1+K_{12})(1-K_{23})e^{+j\beta_2 d}\right], \\ m_{21} &= \frac{1}{4}\left[(1+K_{12})(1-K_{23})e^{-j\beta_2 d} + (1-K_{12})(1+K_{23})e^{+j\beta_2 d}\right], \\ m_{22} &= \frac{1}{4}\left[(1-K_{12})(1-K_{23})e^{-j\beta_2 d} + (1+K_{12})(1+K_{23})e^{+j\beta_2 d}\right]. \end{aligned}$$

# Electromagnetic Analysis of Fabry-Perot Interferometer

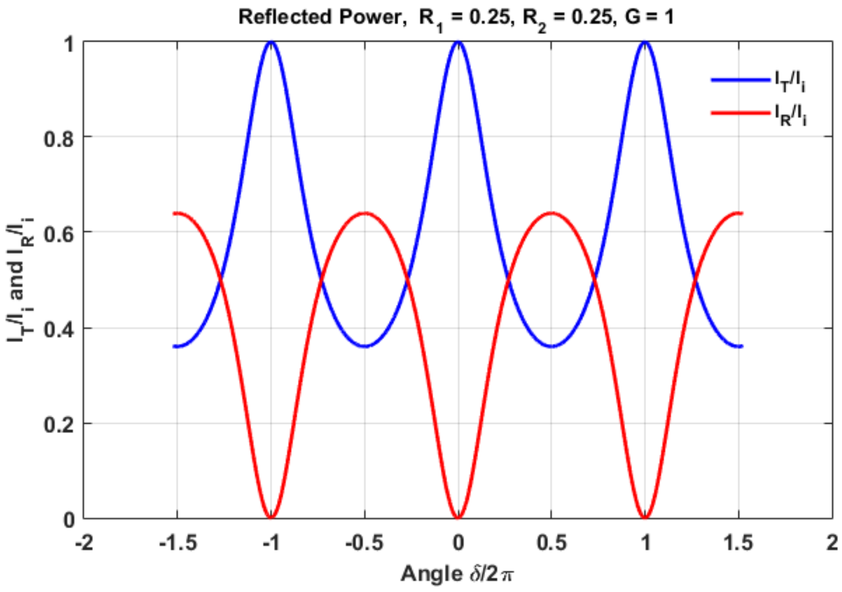


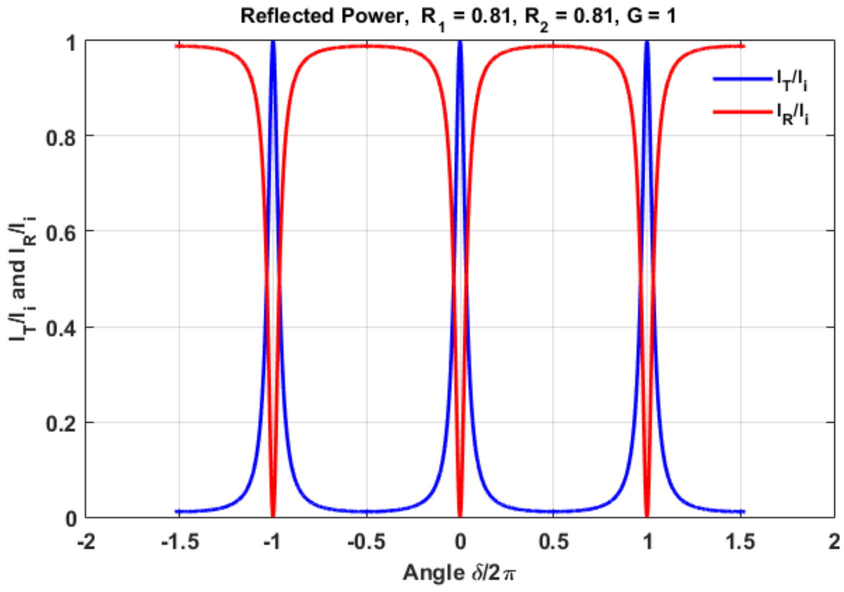

$$\frac{I_r}{I_i} = |r|^2 = \frac{|m_{21}|^2}{|m_{22}|^2},$$

$$\frac{I_t}{I_i} = \frac{n_3\cos\theta_3}{n_1\cos\theta_1}|t|^2 = \frac{n_3\cos\theta_3}{n_1\cos\theta_1}\left[|m_{11}|^2 + \frac{|m_{12}|^2|m_{21}|^2}{|m_{22}|^2} - 2\mathrm{Re}\left\{\frac{m_{11}^* m_{12} m_{21}}{m_{22}}\right\}\right]$$

# Fabry-Perot Interferometer

$$\delta = 2k_0 n_2 d \cos\theta_2$$



# Asymmetric Fabry-Perot Interferometer with Gain or Loss

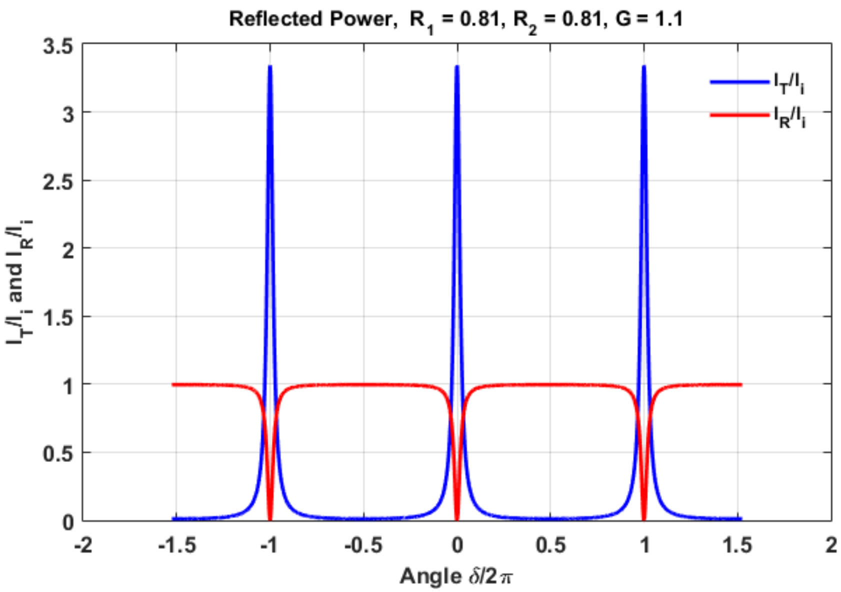
$G = e^{\gamma d}$

$$\frac{I_t}{I_i} = \frac{G(1-R_1)(1-R_2)}{(1-G\sqrt{R_1R_2})^2 + 4G\sqrt{R_1R_2}\sin^2(\delta/2)}$$

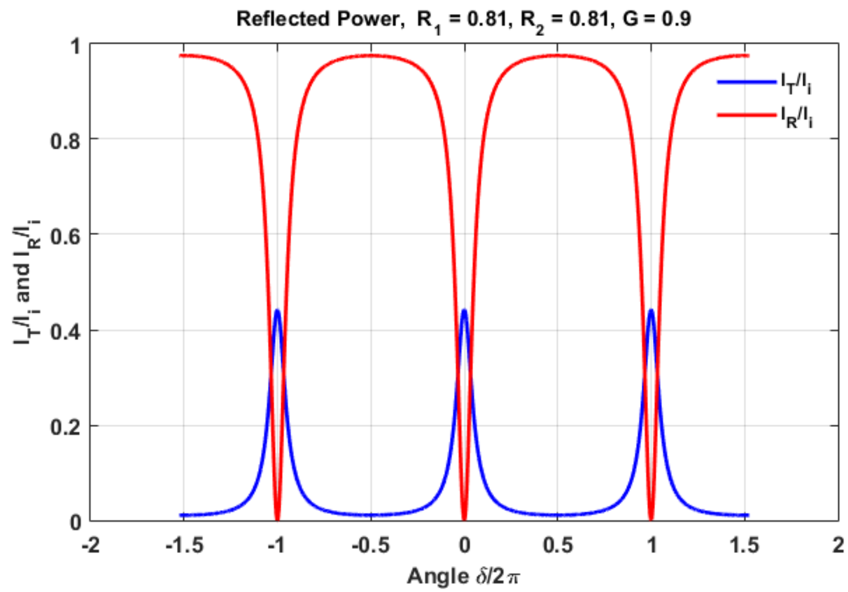
$$\frac{I_r}{I_i} = \frac{(\sqrt{R_1}-\sqrt{R_2})^2 + 4G\sqrt{R_1R_2}\sin^2(\delta/2)}{(1-G\sqrt{R_1R_2})^2 + 4G\sqrt{R_1R_2}\sin^2(\delta/2)}$$

$$\delta = 2k_0 n_2 d\cos\theta_2$$

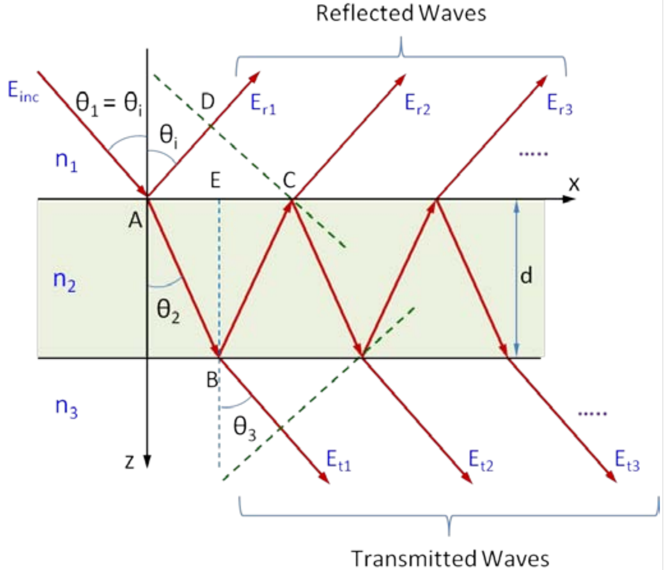
**Gain**



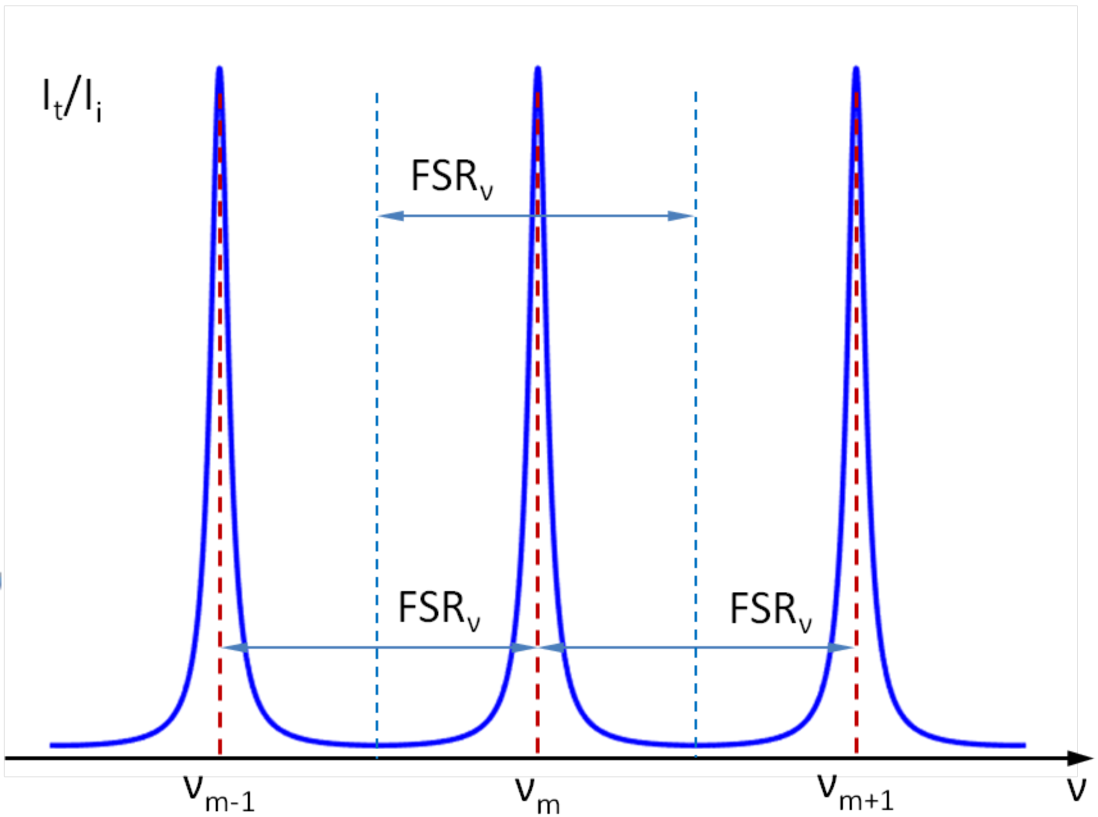

**Loss**



# Symmetric Fabry-Perot Interferometer




## Resonant Frequencies




$$\delta = 2k_0 n_2 d \cos\theta_2$$

$$FSR_\nu = \Delta\nu = \nu_{m+1} - \nu_m = \frac{c}{2dn_2\cos\theta_2}$$

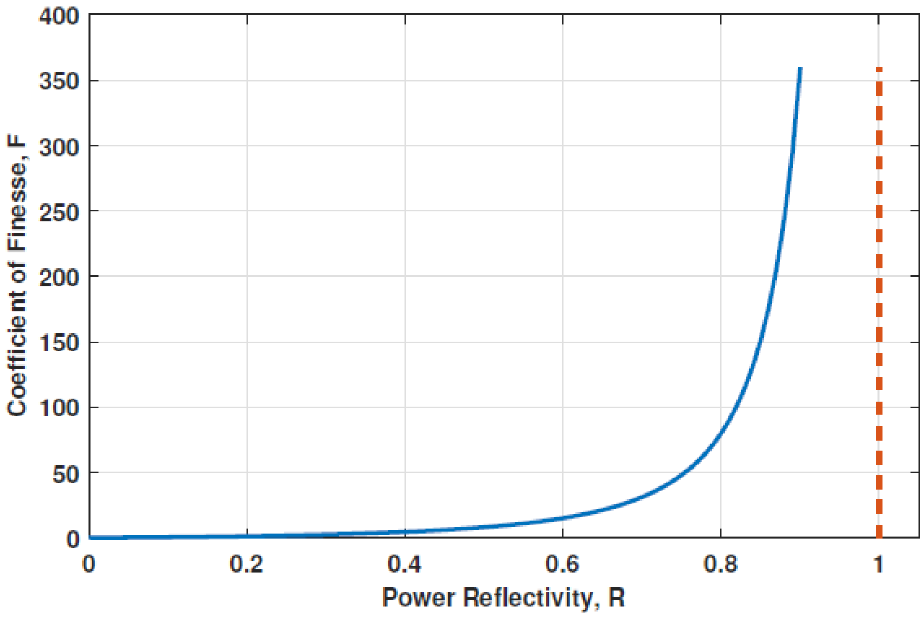
# Symmetric Fabry-Perot Interferometer

$$\delta = 2k_0 n_2 d \cos\theta_2$$

$$\frac{I_t}{I_i} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\delta/2-\phi)} = \frac{1}{1+F\sin^2(\delta/2-\phi)},$$

$$F = \frac{4R}{(1-R)^2}, \qquad \text{Symmetric Fabry-Perot Case.}$$

## Coefficient of Finesse

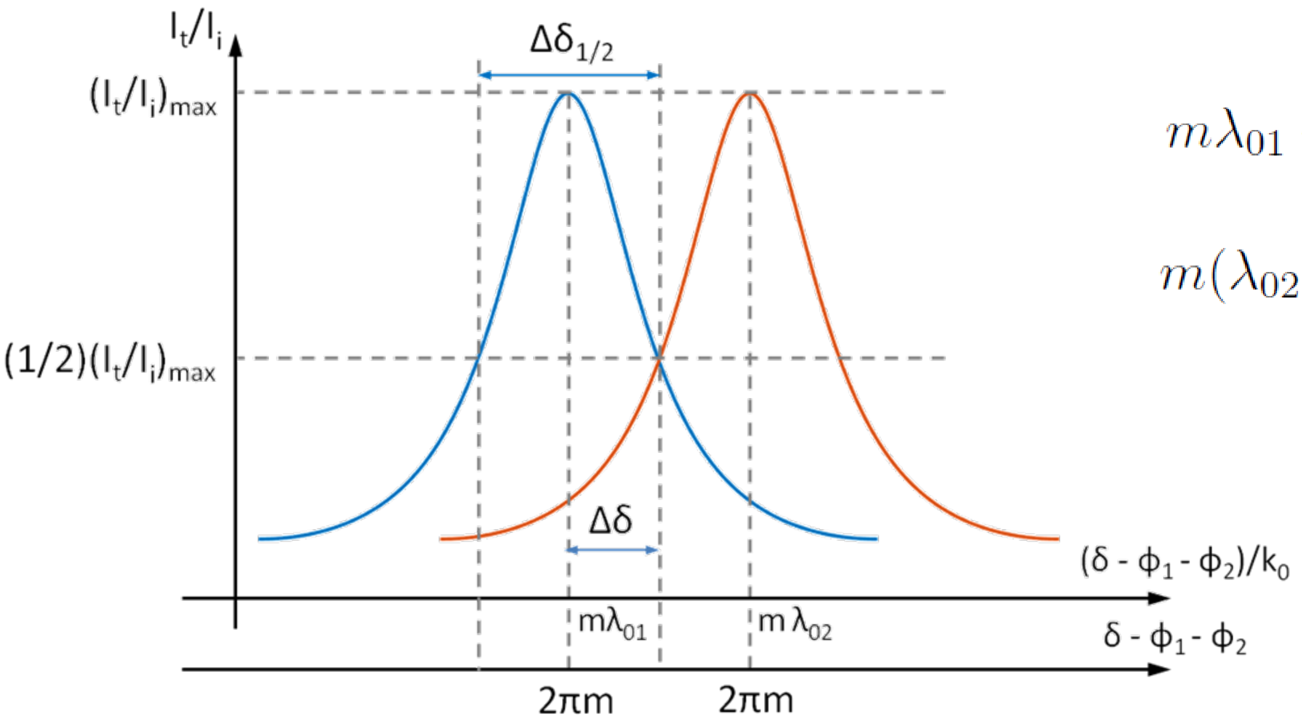



## Coefficient of Finesse

$$F = \frac{4R}{(1-R)^2}$$

# Symmetric Fabry-Perot Interferometer

## Minimum Wavelength Separation



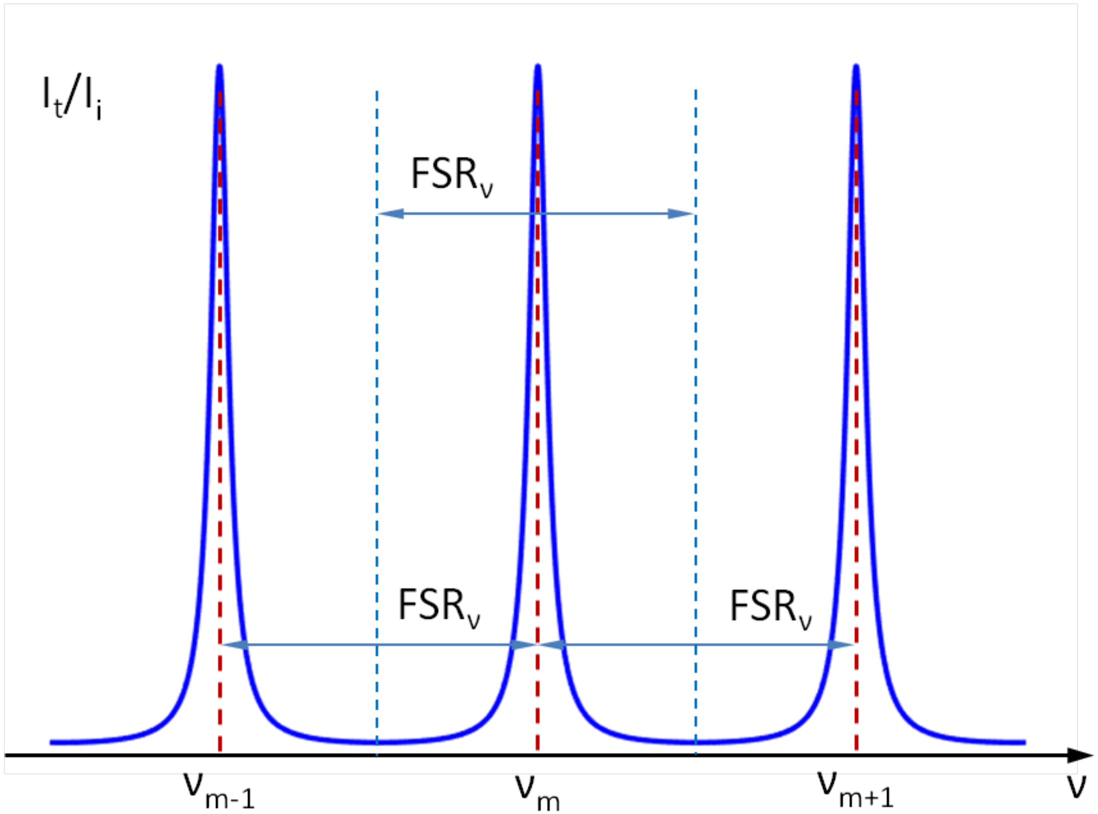

$$m\lambda_{01} + \frac{2\Delta\delta}{k_1} = m\lambda_{02} \Longrightarrow$$

$$m(\lambda_{02} - \lambda_{01}) = \frac{2\lambda_{01}}{\pi\sqrt{F}} \Longrightarrow$$

$$\boxed{\Delta\lambda_{min} = \frac{2\lambda_{01}}{\pi m\sqrt{F}}}$$

$$\frac{I_t}{I_i} = \frac{1}{1 + F\sin^2\left(\frac{2\pi m + \Delta\delta}{2}\right)} = \frac{1}{2} \Longrightarrow \Delta\delta \simeq \frac{2}{\sqrt{F}}$$

# Symmetric Fabry-Perot Interferometer



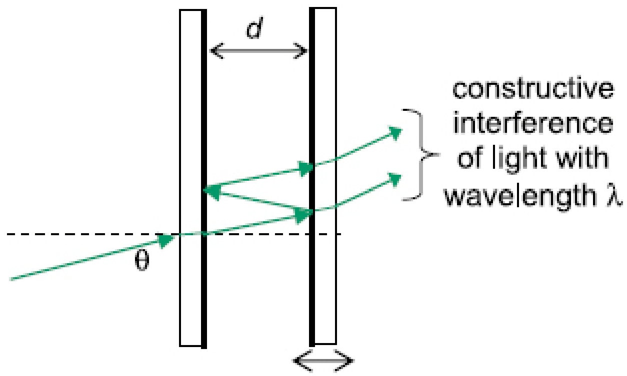

## Resolving Power

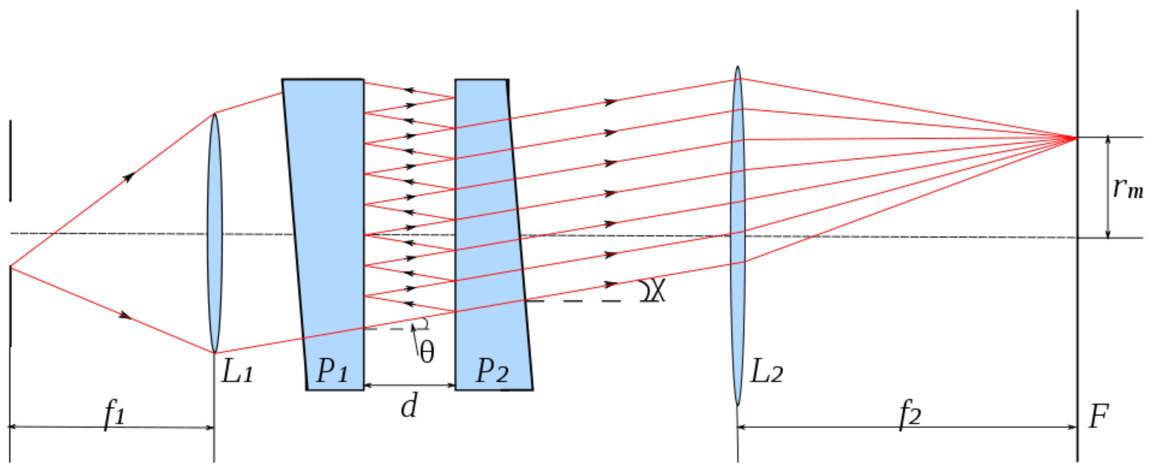
$$\mathcal{R} = \frac{\lambda_0}{\Delta\lambda_{min}} = \frac{\pi m\sqrt{F}}{2}$$

## Finesse

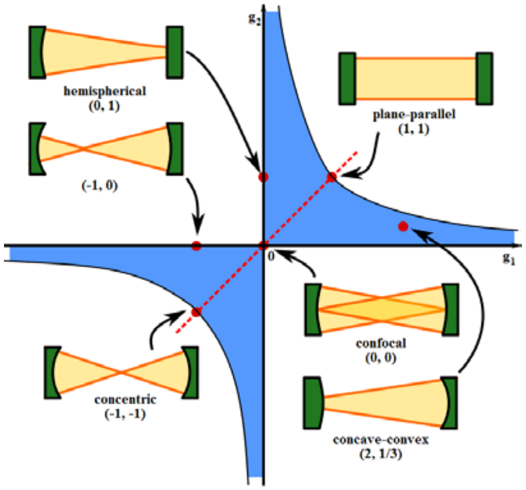
$$\mathcal{F} = \frac{\Delta\lambda_{FSR}}{\Delta\lambda_{min}} = \frac{\pi}{2}\sqrt{F}$$

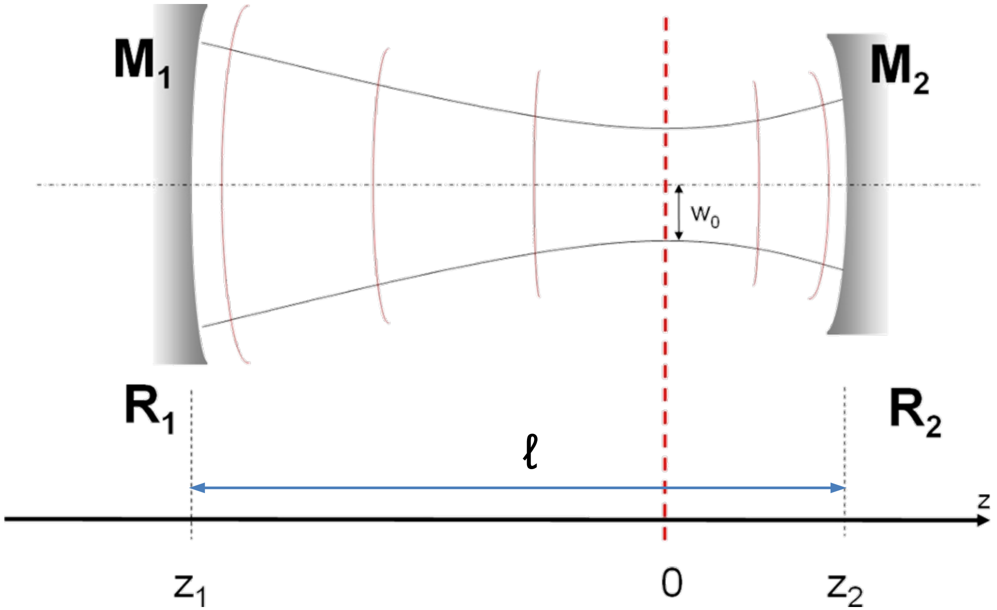
# Fabry-Perot Interferometer



# Two-Mirror Laser Resonator




$$z_1 = -\frac{\ell(R_2-\ell)}{R_1+R_2-2\ell}$$

$$z_2 = +\frac{\ell(R_1-\ell)}{R_1+R_2-2\ell}$$

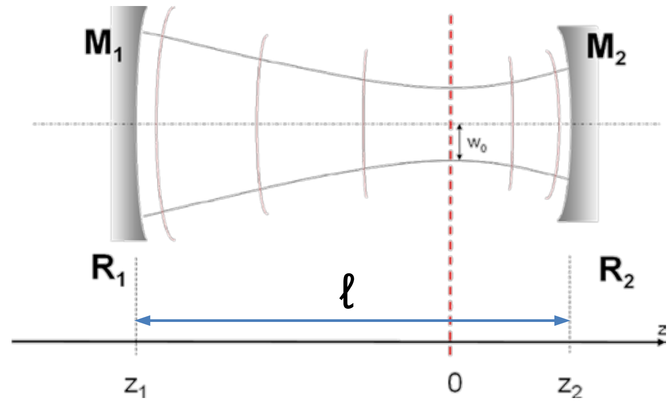
$$z_0^2 = \frac{\ell(R_1-\ell)(R_2-\ell)(R_1+R_2-\ell)}{(R_1+R_2-2\ell)^2}$$

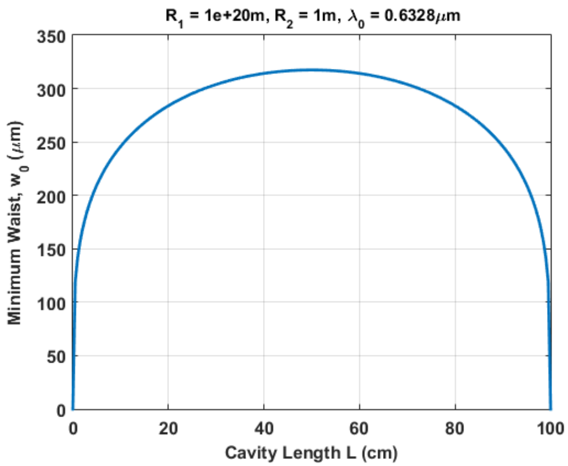
http://www.optique-ingenieur.org/en/courses/OPI_ang_M01_C03/co/Contenu_11.html

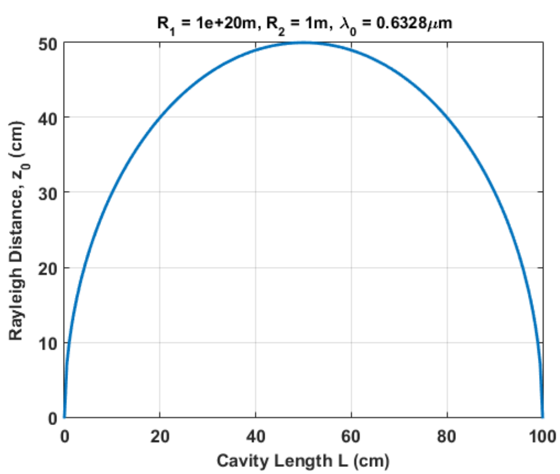
# Two-Mirror Laser Resonator  Example

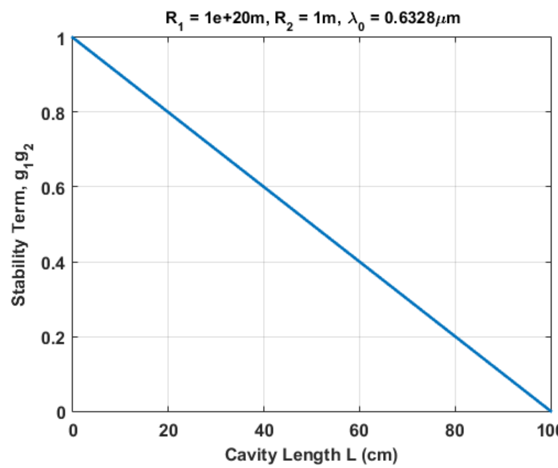
$R_1$ = Infinite

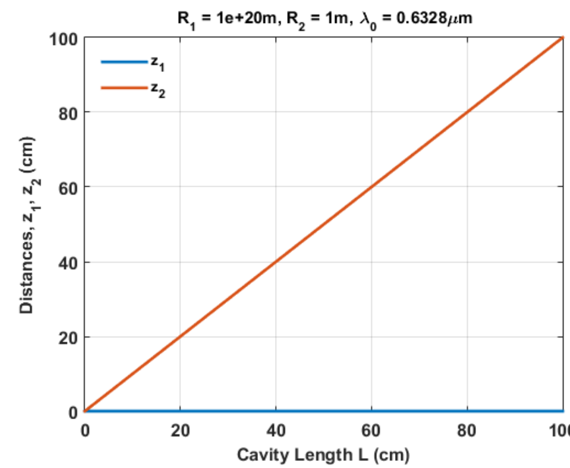
$R_2$ = 1m ($R_2$>0)



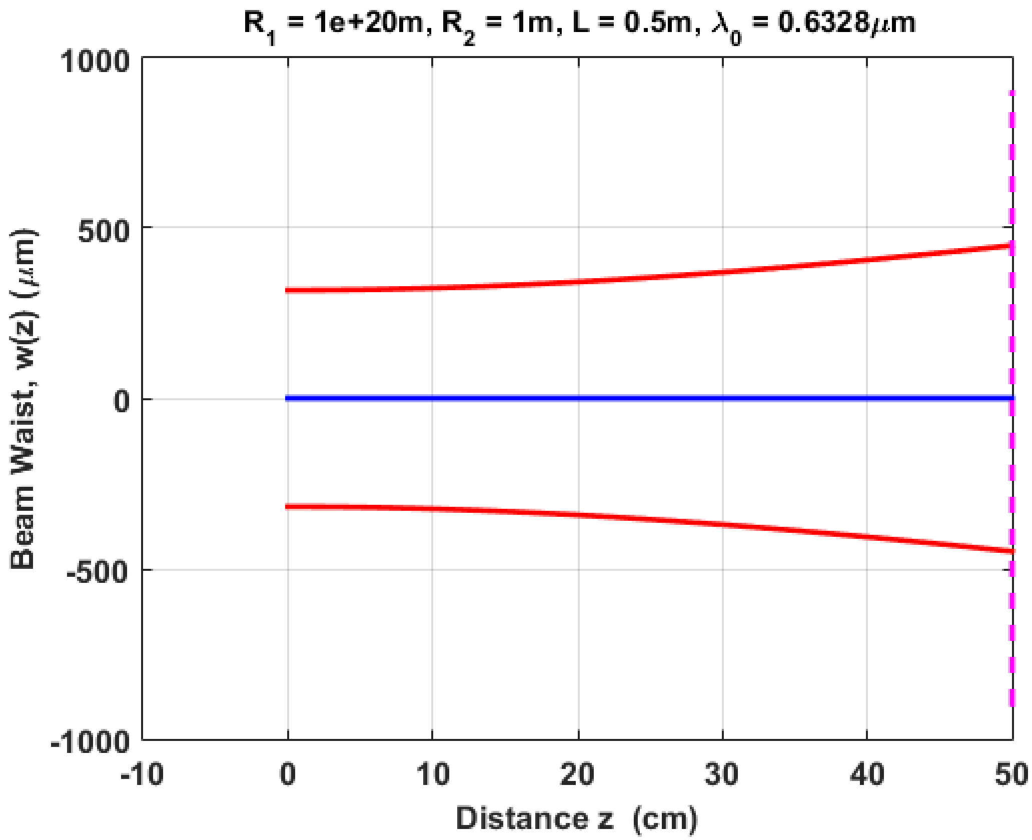



# Two-Mirror Laser Resonator  Example

$R_1$ = Infinite, $R_2$ = 1m (>0), $L = R_2/2$ = 0.5m

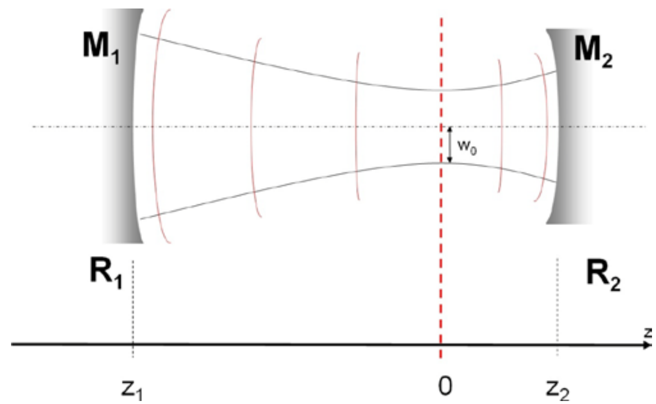


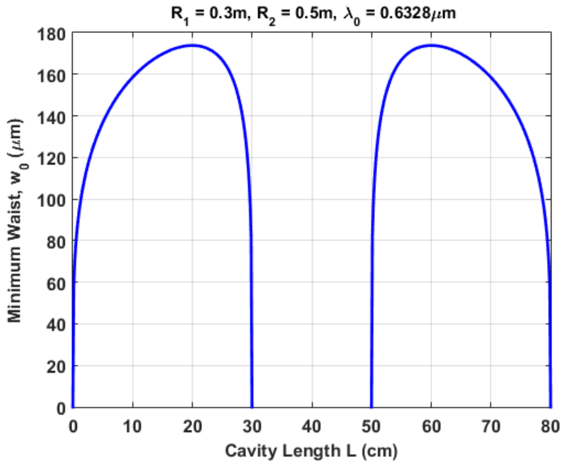

$z_0$ = 0.5m,  $w_0$ = 317.35μm, $z_1$ = 0, $z_2$ = 0.5m

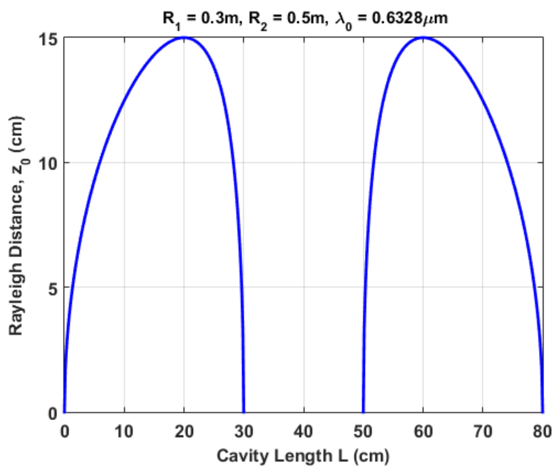
# Two-Mirror Laser Resonator Example

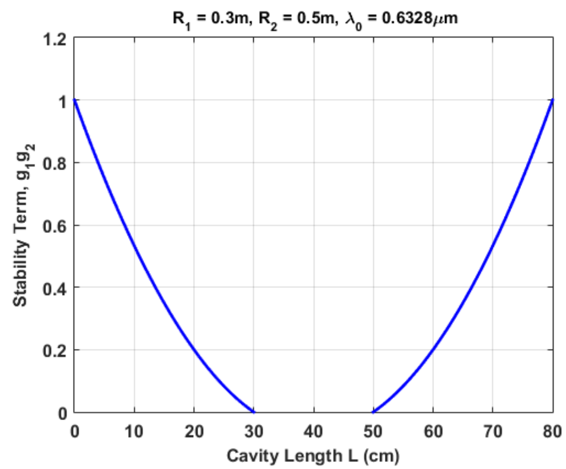
$R_1 = 0.3\text{m}\ (R_1>0)$

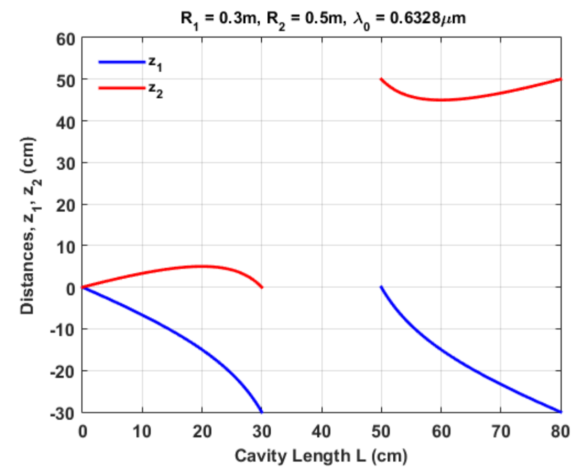
$R_2 = 0.5\text{m}\ (R_2>0)$



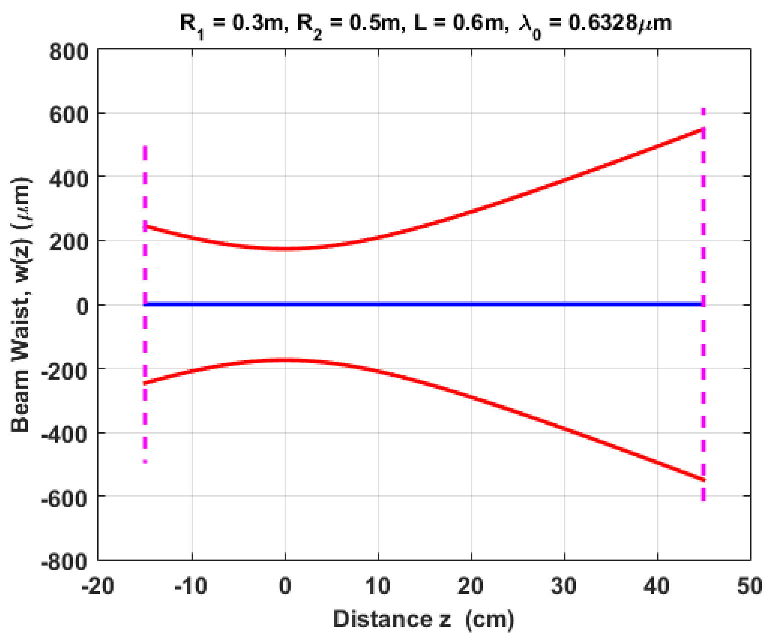



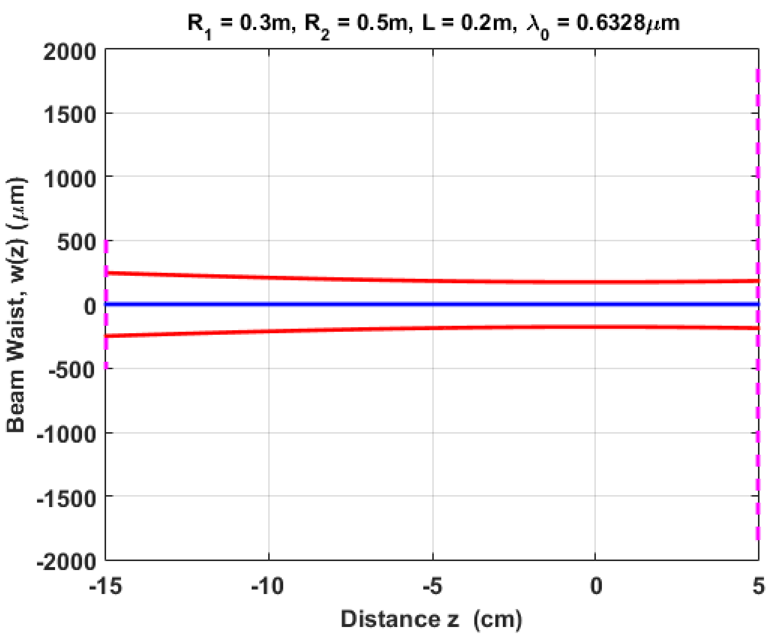
# Two-Mirror Laser Resonator Example

$R_1 = 0.3\text{m}\ (>0),\ R_2 = 0.5\text{m}\ (>0),\ L = 0.6\text{m}$




$z_0 = 0.15\text{m},\ w_0 = 173.82\mu\text{m},$
$z_1 = -0.15\text{m},\ z_2 = 0.45\text{m}$

$R_1 = 0.3\text{m}\ (>0),\ R_2 = 0.5\text{m}\ (>0),\ L = 0.2\text{m}$

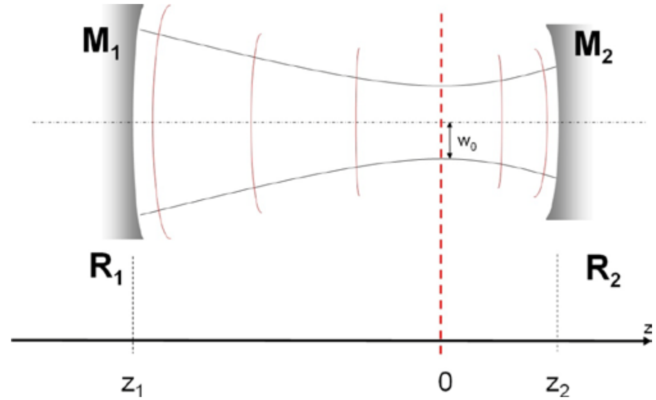


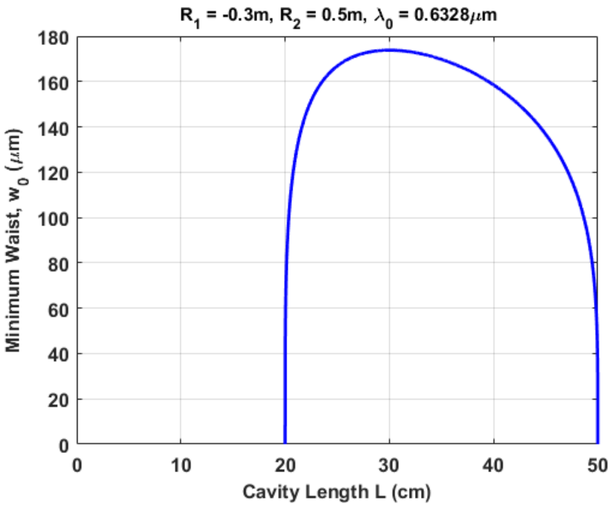

$z_0 = 0.15\text{m},\ w_0 = 173.82\mu\text{m},$
$z_1 = -0.15\text{m},\ z_2 = 0.05\text{m}$

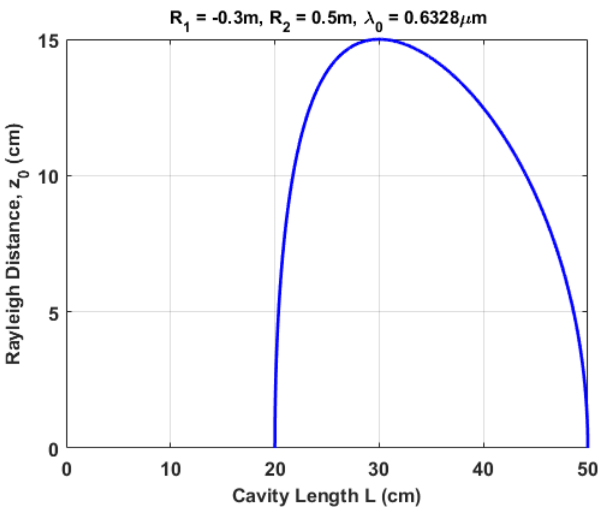
# Two-Mirror Laser Resonator Example

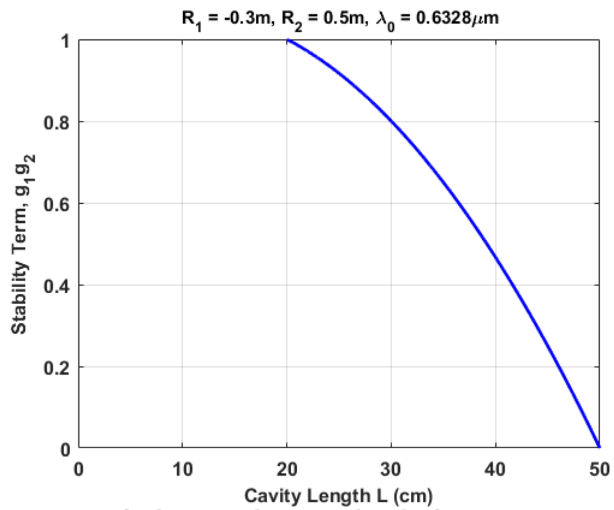
$R_1$ = -0.3m ($R_1$<0)

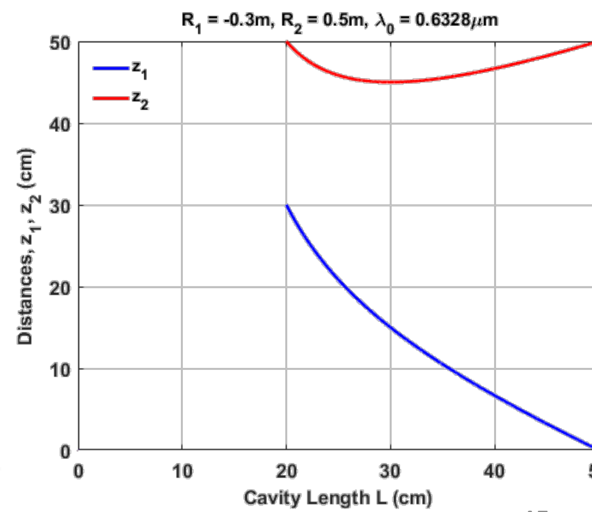
$R_2$ = 0.5m ($R_2$>0)



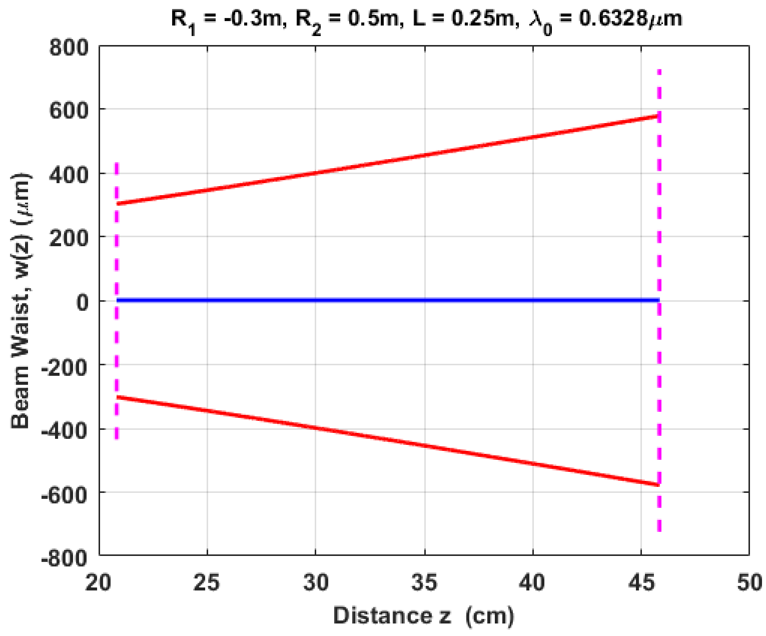



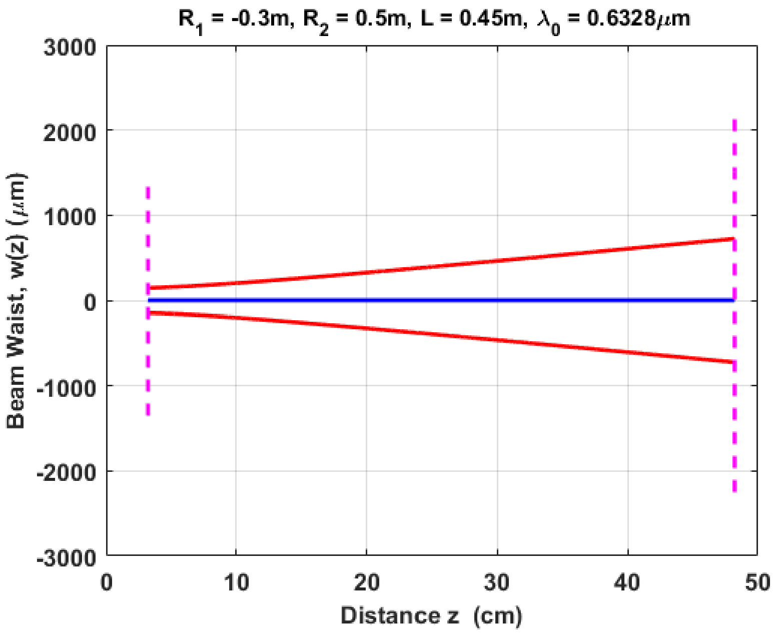
# Two-Mirror Laser Resonator  Example

$R_1$ = -0.3m (<0), $R_2$ = 0.5m (>0), L = 0.25m



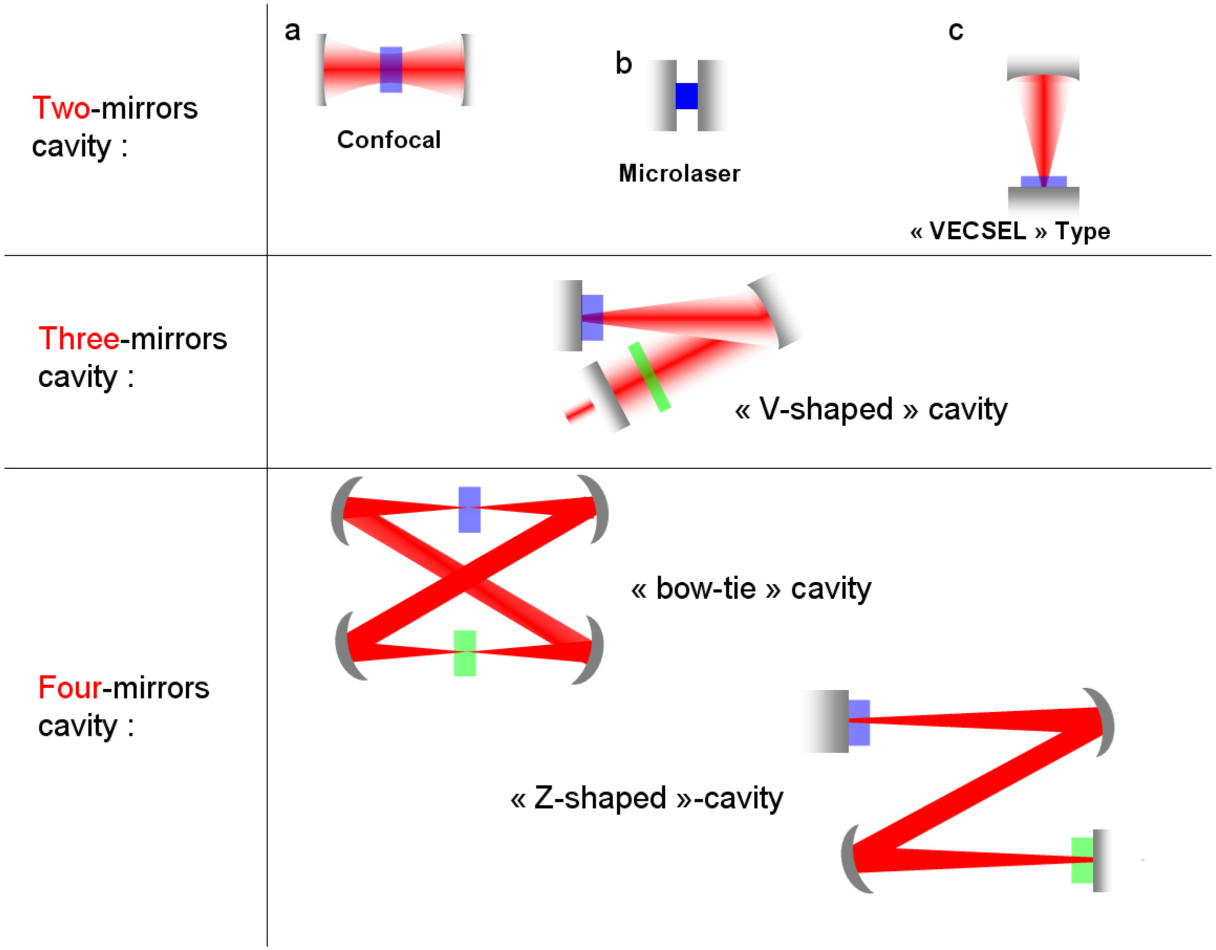

$z_0$ = 0.1382m,  $w_0$ = 166.84μm,
$z_1$ = 0.2083m, $z_2$ = 0.4583m

$R_1$ = -0.3m (<0), $R_2$ = 0.5m (>0), L = 0.5m



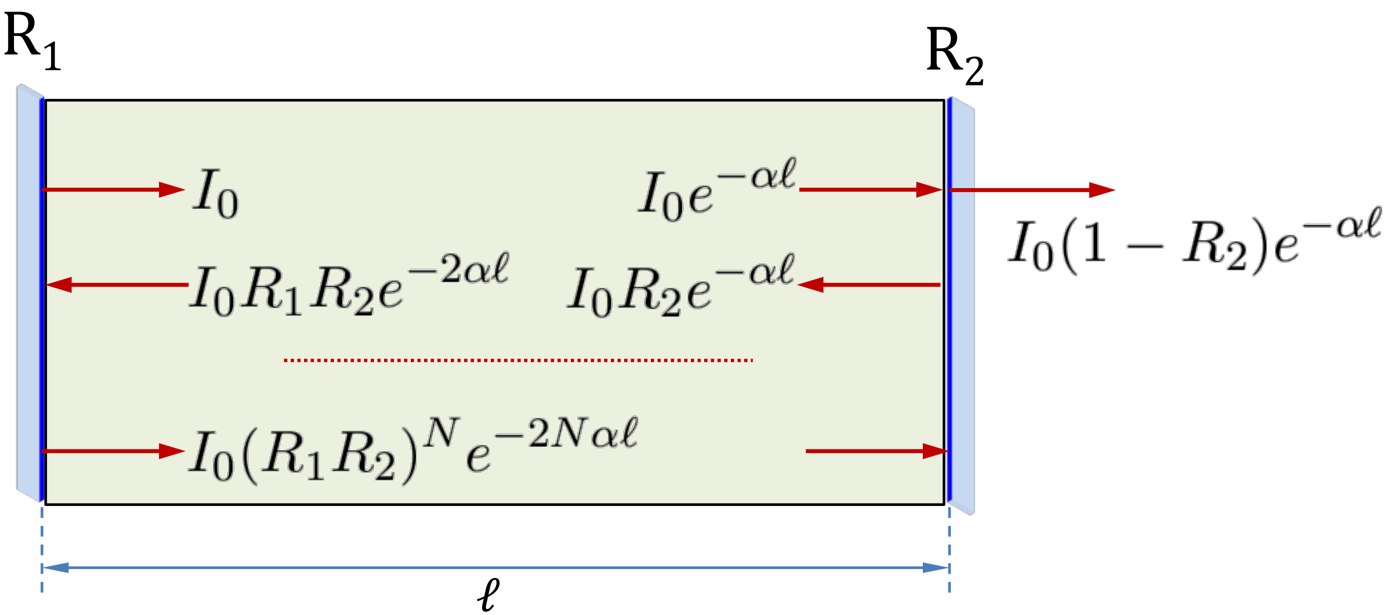

$z_0$ = 0.0928m,  $w_0$ = 136.71μm,
$z_1$ = 0.0321m, $z_2$ = 0.4821m

# Different Laser Resonator Geometries




http://www.optique-ingenieur.org/en/courses/OPI_ang_M01_C03/co/Contenu_12.html

# Cavity Lifetime – Approximate Approach

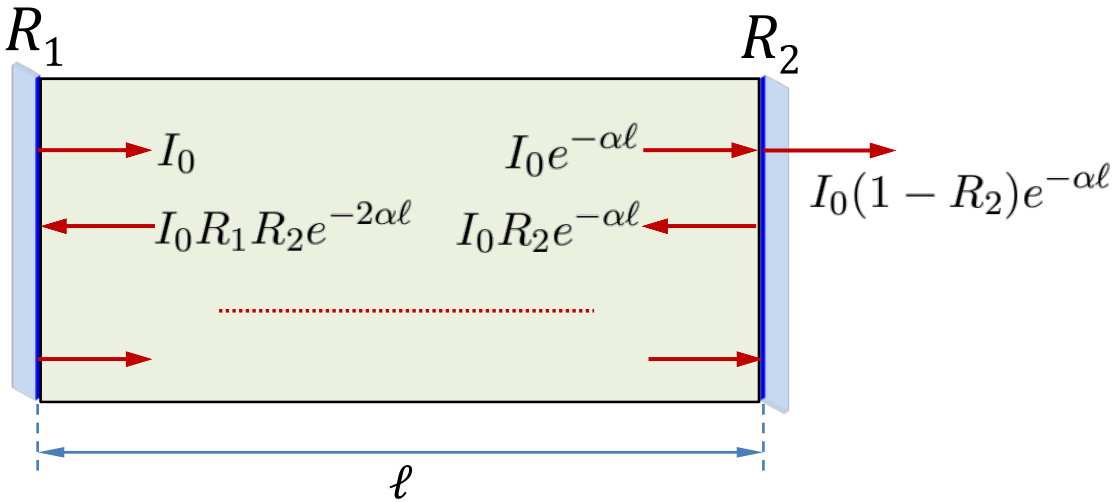



Assume N round-trips in the cavity:

$$\frac{I_0 (R_1 R_2)^N e^{-2N\alpha\ell}}{I_0} = e^{-1} \rightarrow t_N = \frac{2\ell N}{c/n} = t_c$$

$$t_c = \frac{n/c}{\alpha - \frac{1}{\ell}\ln\left(\sqrt{R_1 R_2}\right)}$$

# Cavity Lifetime – Accurate Approach




Rate of decrease in intensity (or number of photons) in the cavity:

$$\frac{dI}{dt} = -\frac{1 - R_1 R_2 e^{-2\alpha\ell}}{\tau_{RT}} I = -\frac{1}{t_c} I$$

$$I(t) = I_0 e^{t/t_c} \longrightarrow \boxed{t_c = \frac{\tau_{RT}}{1 - R_1 R_2 e^{-2\alpha\ell}} = \frac{\tau_{RT}}{1 - S}}$$

Survival Ratio: $S = R_1 R_2 e^{-2\alpha\ell}$

Round-Trip Time: $\tau_{RT} = \dfrac{2\ell}{c/n}$