ΗΛΕΚΤΡΟ-ΟΠΤΙΚΗ ΚΑΙ ΕΦΑΡΜΟΓΕΣ
(ELECTRO-OPTICS)

ΒΑΣΙΚΗ ΘΕΩΡΙΑ ΗΛΕΚΤΡΟ-ΟΠΤΙΚΩΝ ΚΑΙ ΑΚΟΥΣΤΟ-ΟΠΤΙΚΩΝ ΦΑΙΝΟΜΕΝΩΝ

(Fundamentals of Electro-optics and Acousto-optics)

Σημειώσεις

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Electro-optic Effects and Electro-optic Modulation:

For certain crystals it is possible to cause a change of the refractive index with an applied electric field. In addition, this change can be different along the different crystallographic axes and therefore an induced birefringence occurs due to the applied electric field.

For a linear medium the polarization and electric field were related by

$$p(\vec{r},t) = \epsilon_0 \int \tilde{\chi}_e(t-\tau)\, \vec{e}(\vec{r},\tau)\, d\tau \iff \vec{P}(\vec{r},\omega) = \epsilon_0 \chi_e(\omega)\, \vec{E}(\vec{r},\omega).$$

The electro-optic effect is a nonlinear effect. In this case the relation between $\vec{p}$ and $\vec{e}$ can be written as:

$$\vec{p}(\vec{r},t) = \epsilon_0 \int \tilde{\chi}_e^{(1)}(t-\tau_1)\, \vec{e}(\vec{r},\tau_1)\, d\tau_1 + \epsilon_0 \int \tilde{\chi}_e^{(2)}(t-\tau_1, t-\tau_2)\, \vec{e}(\vec{r},\tau_1)\, \vec{e}(\vec{r},\tau_2)\, d\tau_1 d\tau_2 + \epsilon_0 \int \tilde{\chi}_e^{(3)}(t-\tau_1, t-\tau_2, t-\tau_3)\, \vec{e}(\vec{r},\tau_1)\, \vec{e}(\vec{r},\tau_2)\, \vec{e}(\vec{r},\tau_3)\, d\tau_1 d\tau_2 d\tau_3 + \dots$$

where $\chi_e^{(i)}$ $(i=1,2,3)$ are the i-th order susceptibilities. The above equation in the frequency domain is written as:

$$\vec{P}(\vec{r},\omega) = \epsilon_0 \Big[\chi_e^{(1)}(\omega)\, \vec{E}(\vec{r},\omega) + \chi_e^{(2)}(\omega)\Big|_{\omega=\omega_1+\omega_2} \vec{E}(\vec{r},\omega_1)\, \vec{E}(\vec{r},\omega_2) + \chi_e^{(3)}(\omega)\Big|_{\omega=\omega_1+\omega_2+\omega_3} \vec{E}(\vec{r},\omega_1)\, \vec{E}(\vec{r},\omega_2)\, \vec{E}(\vec{r},\omega_3) + \dots$$

It is reminded that $\chi_e^{(i)}$'s are tensor quantities. For example $\chi_e^{(1)}$ is a second-rank tensor (matrix). $\chi_e^{(2)}$ is a third rand tensor, $\chi_e^{(3)}$ is a fourth rank tensor, etc. It is also assumed that $\vec{e}$ comprises of sum of plane waves of frequencies ω_i. The relation between $\tilde{\chi}_e^{(i)}(t)$ and $\chi_e^{(i)}(\omega)$ is:

$$\chi_e^{(i)}(\omega) = \int \tilde{\chi}_e^{(i)}(t-\tau_1, t-\tau_2, \ldots, t-\tau_i)\, e^{-j\omega_1\tau_1 - j\omega_2\tau_2 - \cdots - j\omega_i\tau_i}\, d\tau_1 d\tau_2 \ldots d\tau_i$$

The $\chi_e^{(1)}$ is the linear susceptibility while $\chi_e^{(i)}$ $(i>1)$ are the non-linear susceptibilities. In general the susceptibilities are calculated using a full quantum-mechanical approach. For our purposes it is assumed that $\chi_e^{(i)}$ are known for a given material. Usually, the higher the susceptibility the smaller it is.

For a class of materials $\vec{P}$ can be written as

$$\vec{P}(\vec{r},\omega) = \epsilon_0 \left[\chi_e^{(1)}(\omega)\, \vec{E}(\vec{r},\omega) + \chi_e^{(2)}(\omega)\, \vec{E}(\vec{r},\omega_1)\, \vec{E}(\vec{r},\omega_2) \right]$$

Assuming that $\omega_2 = 0$ (DC applied field) and $\omega_1 = \omega$ (optical field) the above expression is written as

$$\vec{P}(\vec{r},\omega) = \epsilon_0 \left[\chi_e^{(1)}(\omega)\, \vec{E}(\vec{r},\omega) + \chi_e^{(2)}(\omega)\, \vec{E}_0\, \vec{E}(\vec{r},\omega) \right] =$$

$$= \epsilon_0 \left[\chi_e^{(1)}(\omega) + \chi_e^{(2)} \vec{E}_0 \right] \vec{E}(\vec{r},\omega)$$

where $\vec{E}_0$ is the applied DC (or low-frequency) electric field.

Then, $\vec{D}(\vec{r},\omega) = \epsilon_0 \vec{E}(r,\omega) + \vec{P}(\vec{r},\omega) =$

$$= \epsilon_0 \left[1 + \chi_e^{(1)}(\omega) + \chi_e^{(2)}(\omega)\, \vec{E}_0 \right] \vec{E}(\vec{r},\omega)$$

Or $[\varepsilon(\omega)] = 1 + \chi_e^{(1)}(\omega) + \chi_e^{(2)}(\omega)\, \vec{E}_0$. Thus, the relative permittivity tensor depends linearly on the applied electric field $\vec{E}_0$. This dependence of the refractive index on the applied electric field is called "the linear electro-optic effect" or "Pockels" effect. The linear electro-optic effect can be present only in crystals that lack inversion symmetry. This can be easily understood by the following reasoning. The change in the refractive index Δn is related to an applied field $\vec{E}_0$ as $[\Delta n] = [s]\, \vec{E}_0$ where $[s]$ a constant tensor. If the crystal possesses inversion symmetry, then by changing

$\vec{E}_o$ to $-\vec{E}_o$ the same $[\Delta n]$ should be obtained. But in this case $[\Delta n] = -[\Delta n] \sim [\Delta n] = 0 \sim$ a crystal that possesses inversion symmetry cannot exhibit the linear electro-optic effect. Crystals that possess inversion symmetry, however, can exhibit the "quadratic electro-optic effect" or "Kerr effect". In this case $\chi_e^{(2)} \equiv 0$ and $[\varepsilon(\omega)] = 1 + \chi_e^{(1)}(\omega) + \chi_e^{(3)}(\omega)\,\vec{E}_o\vec{E}_o$ which shows the quadratic dependence of the relative permittivity on the applied electric field.

Electro-optic Tensor:

For a biaxial material we showed that the index ellipsoid can be a useful tool for obtaining the eigen-polarizations and their corresponding refractive indices. For this reason it is very convenient to express the electro-optic effects (linear or quadratic) in terms of changes to the index ellipsoid. In the absence of an applied electric field the index ellipsoid can be written in the following form in the principal axes system:

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 .$$

In the presence of an electric field the index ellipsoid can change both dimensions and orientation. Thus, it can be expressed as follows:

$$\left(\frac{1}{n^2}\right)_1 x^2 + \left(\frac{1}{n^2}\right)_2 y^2 + \left(\frac{1}{n^2}\right)_3 z^2 + 2\left(\frac{1}{n^2}\right)_4 yz + 2\left(\frac{1}{n^2}\right)_5 xz + 2\left(\frac{1}{n^2}\right)_6 xy = 1$$

The above equations can also be written as:

$$\vec{x}^T[A]\,\vec{x} = 1 \qquad \vec{E}_o = 0$$

$$\vec{x}^T\left\{[A] + [\Delta(1/n^2)]\right\}\vec{x} = 1 \qquad \vec{E}_o \neq 0$$

where $\vec{x}^T = [x\ y\ z]$ and $[\lambda] = [\varepsilon]^{-1}$ = impermeability tensor ($[\varepsilon]$ is the relative permittivity tensor), and $[\Delta(1/n^2)]$ presents the changes under the influence of the applied electric field.
The induced changes $\Delta(\frac{1}{n^2})_{ij}$ $(i,j = x,y,z)$ are

$$\Delta\left(\frac{1}{n^2}\right)_{ij} = \sum_k r_{ijk} E_{ok} \qquad (i,j,k = x,y,z)$$

for the linear electro-optic effect and

$$\Delta\left(\frac{1}{n^2}\right)_{ij} = \sum\sum g_{ijkl} E_{ok} E_{ol} \qquad (i,j,k,l = x,y,z)$$

for the quadratic electro-optic effect. The r_{ijk} is a third-rank tensor while g_{ijkl} is a fourth-rank tensor. To relate $\Delta(\frac{1}{n^2})_{ij}$ with $\Delta(\frac{1}{n^2})_l$ $(l = 1,2,\dots,6)$ we have to use the reduced subscript notation
For the linear electro-optic effect:

$$\begin{array}{lcl} xx & \rightarrow & 1 \\ yy & \rightarrow & 2 \\ zz & \rightarrow & 3 \\ yz, zy & \rightarrow & 4 \\ xz, zx & \rightarrow & 5 \\ xy, yx & \rightarrow & 6 \end{array} \qquad (r_{ijk} = r_{jik} \text{ due to symmetry of } r_{ijk})$$

and $\Delta(\frac{1}{n^2})_l$'s are given by

$$\begin{bmatrix} \Delta(1/n^2)_1 \\ \Delta(1/n^2)_2 \\ \Delta(1/n^2)_3 \\ \Delta(1/n^2)_4 \\ \Delta(1/n^2)_5 \\ \Delta(1/n^2)_6 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{bmatrix} \begin{bmatrix} E_{ox} \\ E_{oy} \\ E_{oz} \end{bmatrix}$$

Thus, the third-rank electro-optic tensor can be represented by 6×3 matrix.

Similar notation is also valid for the quadratic electro-optic effect. In this case the $\Delta(1/n^2)_l$'s are given by

$$\begin{bmatrix} \Delta(1/n^2)_1 \\ \Delta(1/n^2)_2 \\ \Delta(1/n^2)_3 \\ \Delta(1/n^2)_4 \\ \Delta(1/n^2)_5 \\ \Delta(1/n^2)_6 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} \\ g_{51} & g_{52} & g_{53} & g_{54} & g_{55} & g_{56} \\ g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & g_{66} \end{bmatrix} \begin{bmatrix} E_{ox}^2 \\ E_{oy}^2 \\ E_{oz}^2 \\ E_{oy}E_{oz} \\ E_{ox}E_{oz} \\ E_{ox}E_{oy} \end{bmatrix}$$

where the fourth-rank g_{ijkl} tensor can be represented with a 6×6 matrix. The form of the linear electro-optic and the quadratic electro-optic tensors is shown in the next pages for different material systems.

FORM OF THE LINEAR ELECTRO OPTIC TENSOR

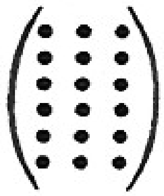
Symbols:

- · zero element
- ● nonzero element
- ● ● equal nonzero elements
- ● ○ equal nonzero elements, but opposite in sign

The symbol at the upper left corner of each tensor is the conventional symmetry group designation.

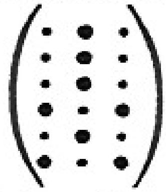
Centrosymmetric—All elements zero

Triclinic

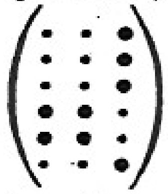


Monoclinic

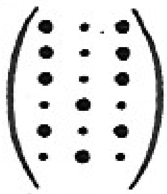
2 (parallel to x_3)



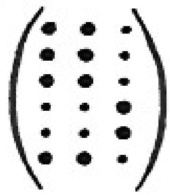
(parallel to x_2)



m (perpendicular to x_3)

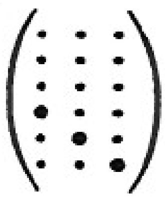


(perpendicular to x_2)

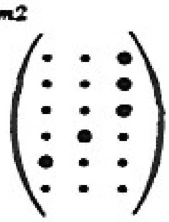


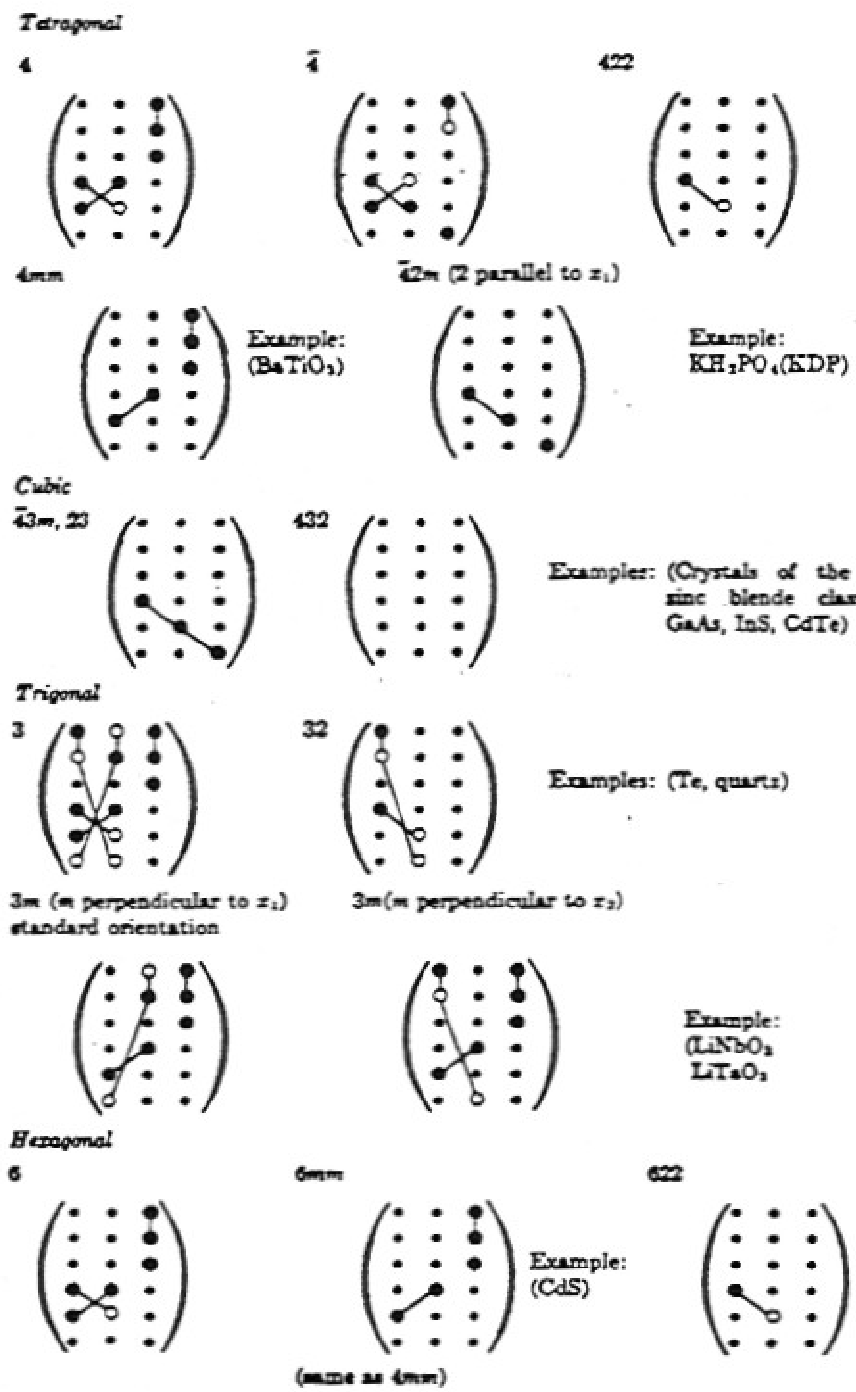
Orthorhombic

222



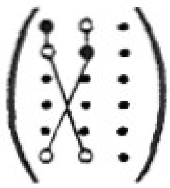
$mm2$



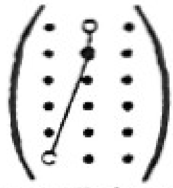


Tetragonal
4
$\bar{4}$
422
4mm
Example: ($BaTiO_3$)
$\bar{4}2m$ (2 parallel to x_1)
Example: KH_2PO_4(KDP)
Cubic
$\bar{4}3m$, 23
432
Examples: (Crystals of the zinc blende class: GaAs, InS, CdTe)
Trigonal
3
32
Examples: (Te, quartz)
3m (m perpendicular to x_1) standard orientation
3m(m perpendicular to x_2)
Example: ($LiNbO_3$ $LiTaO_3$
Hexagonal
6
6mm
Example: (CdS)
622
(same as 4mm)

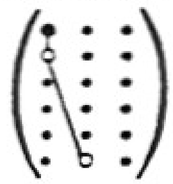
$\bar{6}$



$\bar{6}m2$ (m perpendicular to x_1 standard orientation)



(m perpendicular to x_2)



FORM OF THE QUADRATIC ELECTRO OPTIC TENSOR

KEY TO NOTATION

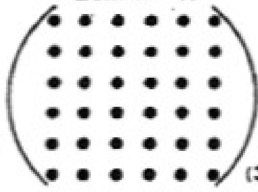
In both π and p matrices

- · zero component
- • non-zero component
- •—• equal components
- •—o components numerically equal, but opposite in sign.

In the p matrices

- ◉ a component equal to the heavy dot component to which it is joined
- ◉ a component equal to minus the heavy dot component to which it is joined
- x $\frac{1}{2}(p_{11}-p_{12})$.

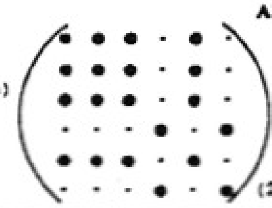
TRICLINIC
Both classes



(36)

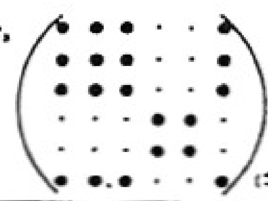
MONOCLINIC
All classes

Diad ∥ x_2 (standard orientation)



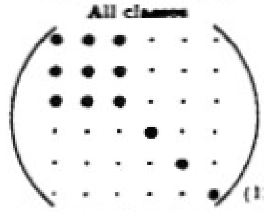
(20)

Diad ∥ x_3

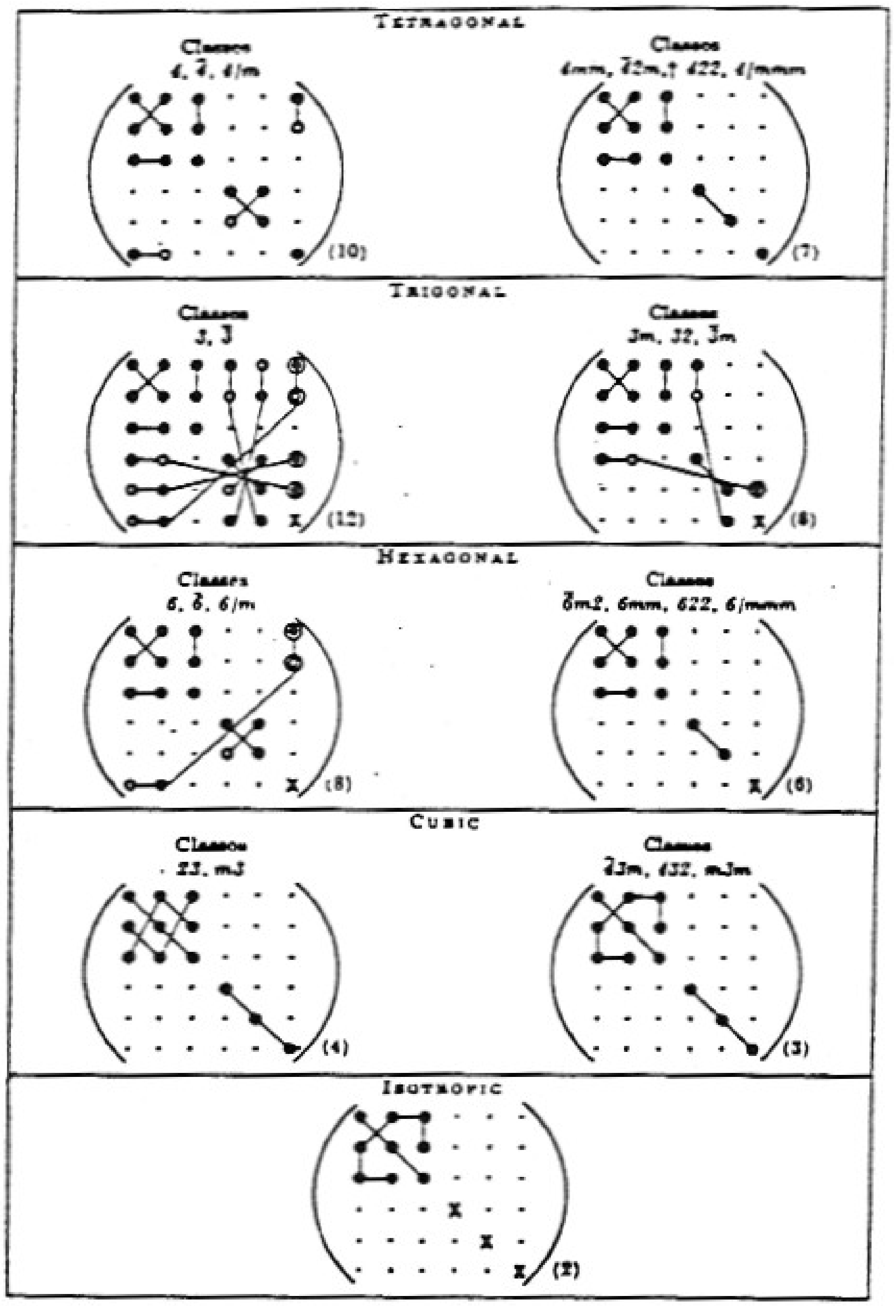


(20)

ORTHORHOMBIC
All classes



(12)


TETRAGONAL
Classes
4, 4̄, 4/m
(10)
Classes
4mm, 4̄2m,† 422, 4/mmm
(7)
TRIGONAL
Classes
3, 3̄
(12)
Classes
3m, 32, 3̄m
(6)
HEXAGONAL
Classes
6, 6̄, 6/m
(8)
Classes
6̄m2, 6mm, 622, 6/mmm
(6)
CUBIC
Classes
23, m3
(4)
Classes
4̄3m, 432, m3m
(3)
ISOTROPIC
(2)

Example in KDP crystal:

For the KDP crystal the (linear) electro-optic effect tensor is:

$$[r_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{bmatrix} \qquad (r_{52} = r_{41})$$

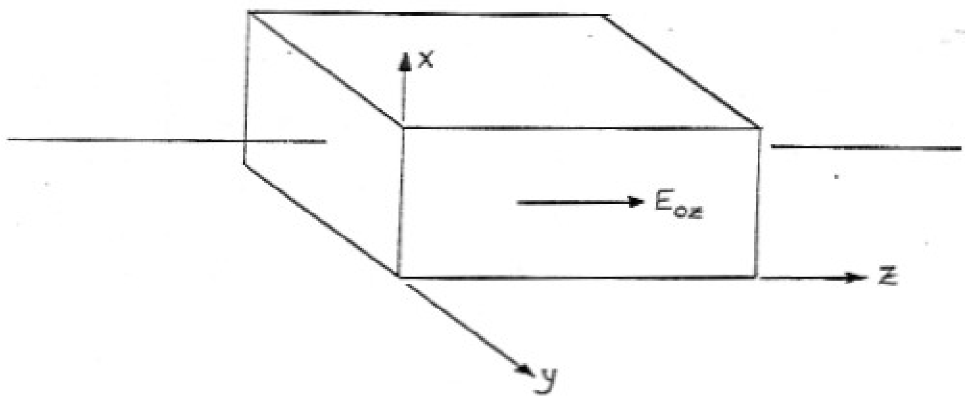
(expressed in the principal axes system)

The relative permittivity and impermeability tensors are:

$$[\varepsilon_r] = \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_E^2 \end{bmatrix} \qquad [\eta] = [\varepsilon_r]^{-1} = \begin{bmatrix} 1/n_o^2 & 0 & 0 \\ 0 & 1/n_o^2 & 0 \\ 0 & 0 & 1/n_E^2 \end{bmatrix}$$

(uniaxial crystal).

Assume the following configuration:



The index ellipsoid expressed in the principal axes system (with $E_{oz}=0$) is:

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_E^2} = 1$$

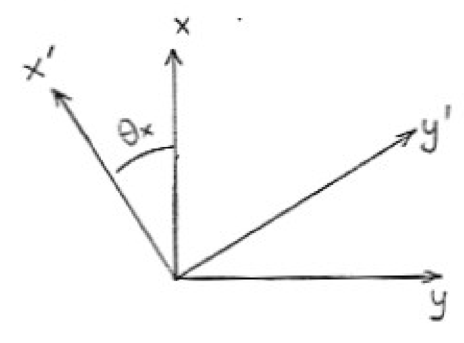
The application of the electric field E_{oz} along the z-direction causes the following $\Delta(1/n^2)$ changes:

$$
\begin{bmatrix} \Delta(1/n^2)_1 \\ \Delta(1/n^2)_2 \\ \Delta(1/n^2)_3 \\ \Delta(1/n^2)_4 \\ \Delta(1/n^2)_5 \\ \Delta(1/n^2)_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_{0z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_{63} E_{0z} \end{bmatrix}
$$

Therefore, the index ellipsoid after the application of the electric field becomes:

$$
\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_E^2} + 2 r_{63} E_{0z}\, xy = 1 .
$$

The cross-product term xy cause the index ellipsoid to rotate. We need to transform it to a new principal axes system. Since only the xy term appears the new principal axes system (x', y', z') will be related with the old one (x, y, z) through a rotation about the z axis. Assume that the rotation angle is θ_x.



$$
\begin{aligned}
x &= x'\cos\theta_x + y'\sin\theta_x \\
y &= -x'\sin\theta_x + y'\cos\theta_x \\
z &= z'
\end{aligned}
$$

Replacing x, y, z in the index-ellipsoid equation with $x', y',$ and z' we get:

$$
x'^2\left(\frac{1}{n_o^2} - r_{63} E_{0z} \sin 2\theta_x\right) + y'^2\left(\frac{1}{n_o^2} + r_{63} E_{0z} \sin 2\theta_x\right) + z'^2\, {}^1\!/_{n_E^2} +
$$

$$
x'y'\,[\cos 2\theta_x]\, 2 r_{63} E_z = 1
$$

The required rotation angle θ_x is such that the coefficient of the cross-product term should be zero. For this case $\cos 2\theta_x = 0 \Rightarrow$ $\theta_x = \pi/4 = 45°$. Then the index ellipsoid becomes:

$$x'^2\left(\frac{1}{n_o^2} - r_{63}E_{0z}\right) + y'^2\left(\frac{1}{n_o^2} + r_{63}E_{0z}\right) + z'^2\frac{1}{n_E^2} = 1.$$

Actually, after the application of the electric field the crystal became biaxial. The new principal refractive indices are:

$$\frac{1}{n_{x'}^2} = \frac{1}{n_o^2} - r_{63}E_{0z}, \quad \frac{1}{n_{y'}^2} = \frac{1}{n_o^2} + r_{63}E_{0z}, \quad \frac{1}{n_{z'}^2} = \frac{1}{n_E^2}.$$

$$n_{x'}^2 = \frac{1}{\frac{1}{n_o^2}(1 - n_o^2 r_{63}E_{0z})} \Rightarrow n_{x'} = n_o \frac{1}{\sqrt{1 - n_o^2 r_{63}E_{0z}}} \Rightarrow$$

$$n_{x'} \simeq n_o\left[1 + \tfrac{1}{2}n_o^2 r_{63}E_{0z}\right] = n_o + \frac{n_o^3}{2}r_{63}E_{0z}$$ (assuming that $n_o^2 r_{63}E_{0z} \ll 1$ which is generally true). Similarly,

$n_{y'} \simeq n_o - \frac{n_o^3}{2}r_{63}E_{0z}$ and of course $n_{z'} = n_E$.

Therefore, for different directions of propagation and polarizations of the electric field we can compute the corresponding refractive indices.

For example, for propagation in the z-direction:

a) for polarization along x' $\rightarrow n = n_{x'} = n_o + \frac{n_o^3}{2}r_{63}E_{0z}$ (slow axis)

b) for polarization along y' $\rightarrow n = n_{y'} = n_o - \frac{n_o^3}{2}r_{63}E_{0z}$ (fast axis)

General Solution:

The index ellipsoid can be represented with the quadric form:

$$\vec{x}^T[A]\,\vec{x} = 1$$ where $[A]$ is the impermeability tensor ($[A]=[\varepsilon_r]^{-1}$). The above equation can represent any form of the index ellipsoid after the application of an external electric field.

In order to find the new principal axes system we have to diagonalize the impermeability matrix. Each of the new principal axes should be normal to the index ellipsoid. In other words if $\vec{v}$ represents one of the new principal axes then $\vec{\nabla} f\big|_{\vec{v}} = \lambda'\vec{v}$ where $f = f(\vec{x}) = \vec{x}^T[A]\vec{x} - 1 = 0 \rightsquigarrow \vec{\nabla} f\big|_{\vec{v}} = 2[A]\vec{v}$ and then we need to solve the following eigenvalue-eigenvector problem:

$$[A]\vec{v} = \underbrace{\lambda'}_{2}\vec{v} = \lambda\vec{v} \rightarrow \big[[A] - \lambda[I]\big]\,\vec{v} = 0.$$

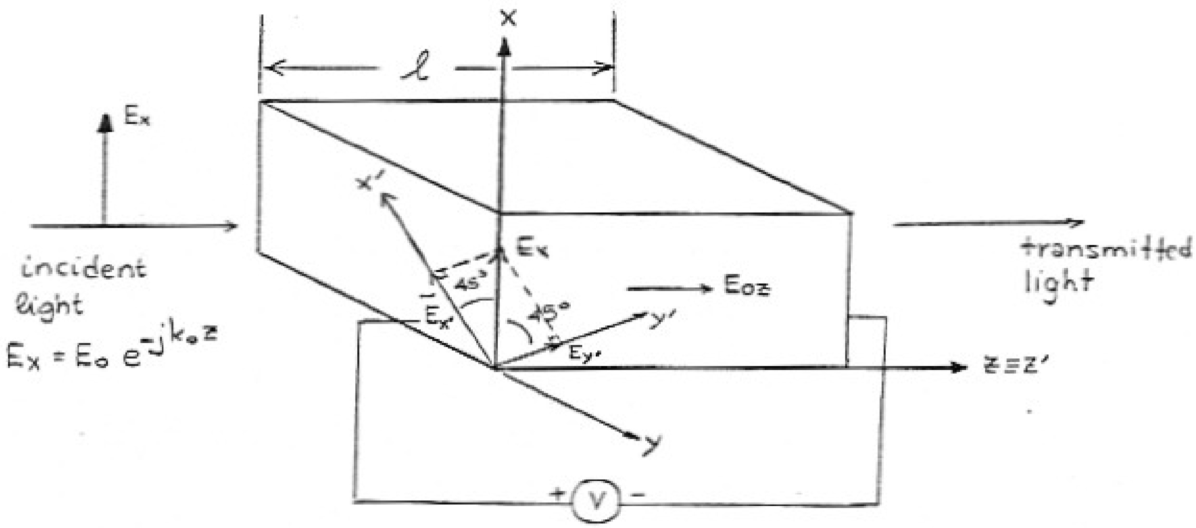
Consequently, λ is an eigenvalue of $[A]$ and $\vec{v}$ is an eigenvector. Since $[A]$ is symmetric, the three eigenvectors are mutually orthogonal. Since $f(\vec{v}) = 0 \Rightarrow \vec{v}^T[A]\vec{v} = 1 \Rightarrow \vec{v}^T\lambda\vec{v} = 1 \Rightarrow \lambda|\vec{v}|^2 = 1 \Rightarrow |\vec{v}|^2 = 1/\lambda \Rightarrow |\vec{v}| = 1/\sqrt{\lambda}$ (thus, $[A]$ should be positive definite which is implied for an ellipsoid). But the magnitude of the new principal axes correspond to the new principal refractive indices. Therefore, $n = |\vec{v}| = 1/\sqrt{\lambda}$.

It would be very instructive to solve the previous example using the above procedure. You can do that as an exercise.

For more information on electro-optic effect calculations see

T. A. Maldonado and T. K. Gaylord, "Electro-optic effect calculations: simplified procedure for arbitrary cases," Appl. Opt., vol. 27, pp. 5051-5066, Dec. 15, 1988.

Electro-optic Retardation:



Assume the example with the KDP crystal as shown again in the figure. The new principal axis system is denoted by $x'y'z'$. Assume an incident plane wave with polarization along the x-axis. Inside the crystal the electric field can be decomposed into the $E_{x'}$ and $E_{y'}$ components which are parallel to the x', y' new principal axes.

Then, $(k_o = 2\pi/\lambda_o)$

$$E_{x'} = \frac{\sqrt{2}}{2} E_o \, e^{-jk_o n_{x'} z} = \frac{\sqrt{2}}{2} E_o \, e^{-jk_o [n_o + \frac{n_o^3}{2} r_{63} E_{oz}] z}$$

$$E_{y'} = \frac{\sqrt{2}}{2} E_o \, e^{-jk_o n_{y'} z} = \frac{\sqrt{2}}{2} E_o \, e^{-jk_o [n_o - \frac{n_o^3}{2} r_{63} E_{oz}] z}$$

Since $E_{x'}$, $E_{y'}$ see two different refractive indices, a phase difference between them will built up. At the end of the crystal $(z=\ell)$ the phase difference $\Gamma = \phi_{x'} - \phi_{y'} = k_o n_o^3 r_{63} E_{oz} \, \ell = k_o n_o^3 r_{63} \frac{V}{\ell} \ell = k_o n_o^3 r_{63} V$

This phase difference is called retardation and at the output of the crystal the electric field phasor is:

$$\vec{E} = E_{x'} \hat{x}' + E_{y'} \hat{y}' = \frac{\sqrt{2}}{2} E_o \left[e^{-j\overbrace{(\phi_{x'} - \phi_{y'})}^{\Gamma}} \hat{x}' + \hat{y}' \right] e^{-jk_o z}$$

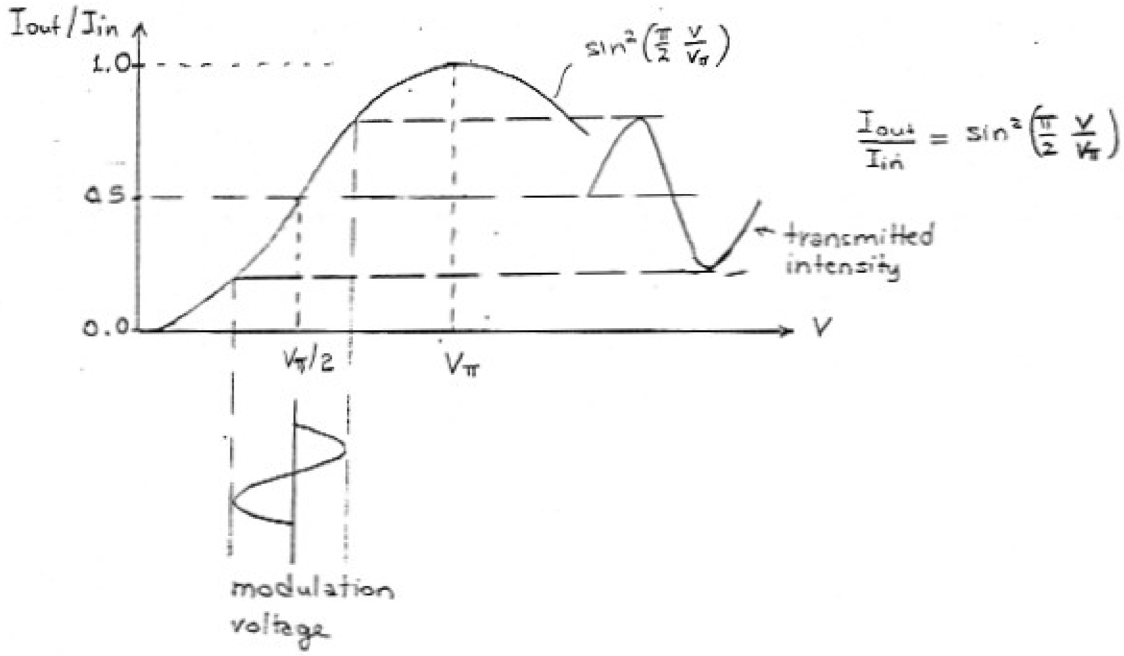
Depending on Γ the polarization of the output wave can be linear, circular, or elliptical.

An electro-optic amplitude modulator can be implemented if two polarizers are used in conjunction with the crystal. The input polarizer is along the x-direction while the output polarizer is along the y-direction. Then, the relation between the input and the output light intensity is

$$\frac{I_{out}}{I_{in}} = \sin^2 \frac{\Gamma}{2} \qquad \Gamma = \frac{2\pi}{\lambda_0} n_o^3 r_{63} V = \pi \frac{V}{V_\pi}$$

where $V_\pi = \dfrac{\lambda_0}{2 r_{63} n_o^3} \doteq$ half-wave voltage.

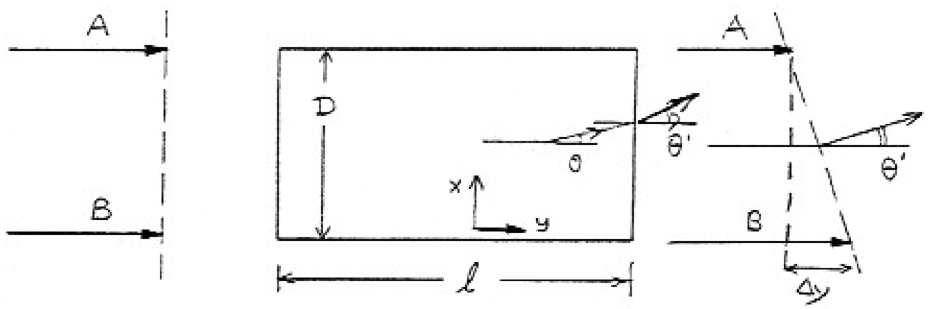
The modulation can be illustrated by the following diagram.



In the above diagram a constant phase shift of $\frac{\pi}{2}$ is achieved through a "quarter-wave" plate in order to linearize the response of the modulator. The $\Gamma = \Gamma(t) = \frac{\pi}{2} + \Gamma_m \sin \omega_m t$

Without using any output polarizer or quarter-wave plate we can directly modulate the phase (phase modulation) if we use one of the eigenpolarizations incident on the crystal.

Electro-optic Beam Deflection:



Assume an electro-optic crystal with an induced variation of the refractive index $n(x) = n + \frac{\Delta n}{D} x$. Ray A traverses the crystal in a time

$T_A = \frac{\ell}{c}(n+\Delta n)$ and ray B in a time

$T_B = \frac{\ell}{c} n$.

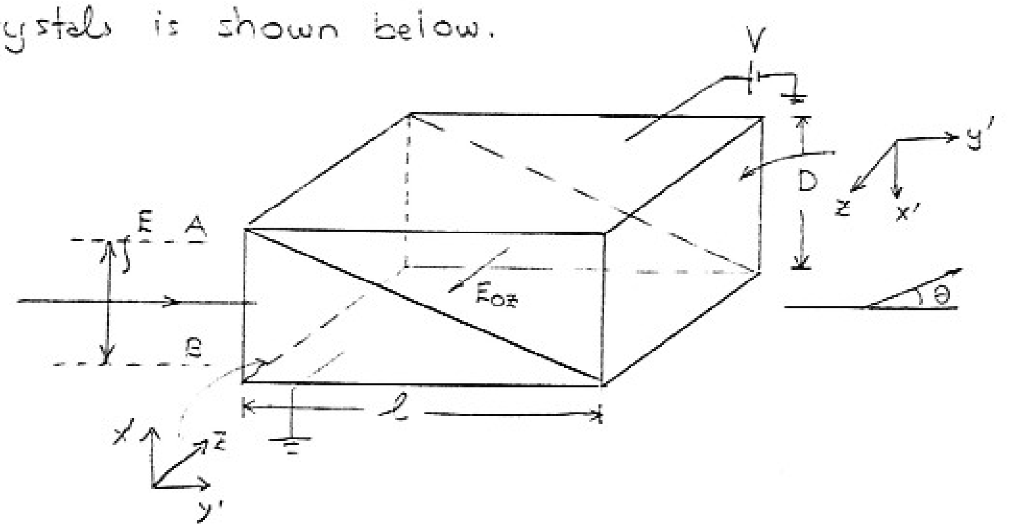
The difference Δy that rey B is leading ray A is $\Delta y = \frac{c}{n}(T_A - T_B) = \ell \frac{\Delta n}{n}$. This difference is going to tilt the phase-front of the wave by an angle θ' (inside the crystal), where θ is

$\theta' \simeq \tan\theta' = \frac{\Delta y}{D}$ (positive as shown). The angle θ outside the crystal is related to θ' via Snell's law. Therefore,

$n \sin\theta' = 1.0 \sin\theta = \sin\theta$, ($n_{out} = 1.0$), and since θ, θ' are small

$\theta = n\theta'$. Thus, $\theta = n \, \ell \frac{\Delta n}{n} \frac{1}{D} = \ell \frac{dn}{dx}$ where $\frac{\Delta n}{D} = \frac{dn}{dx}$

A simple realization of this electro-optic deflector using KDP crystals is shown below.



This electro-optic deflector is comprised of two KDP crystals with opposite z-axes (the x',y' are the principal axes after the application of the electric field along the z-axis).

The A ray sees an index $n_A = n_o + \frac{n_o^3}{2} r_{63} E_{oz}$ while the B ray sees an index $n_B = n_o - \frac{n_o^3}{2} r_{63} E_{oz}$.

$\Delta n = n_A - n_B = n_o^3 r_{63} E_{oz}$ and the deflection angle is

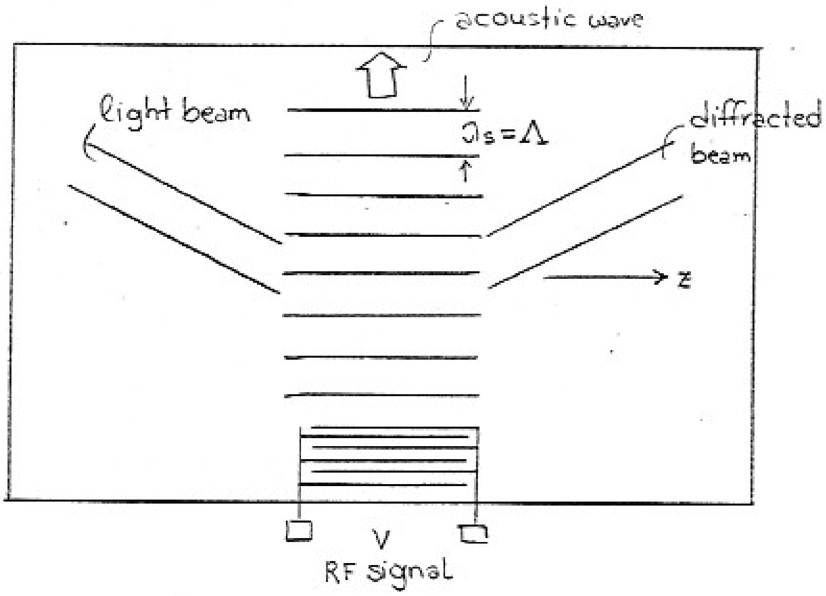
$$\theta = \ell \cdot \frac{\Delta n}{D} = \frac{\ell}{D} n_o^3 r_{63} E_{oz}$$

Interaction of Light and Sound

A sound wave propagating inside a crystal is a strain wave that travels with the velocity of sound. This strain wave produces a change of the refractive index in the crystal. For an acoustic wave traveling in the z-direction in the crystal with a sound velocity v_s, the induced refractive index change can be written as:

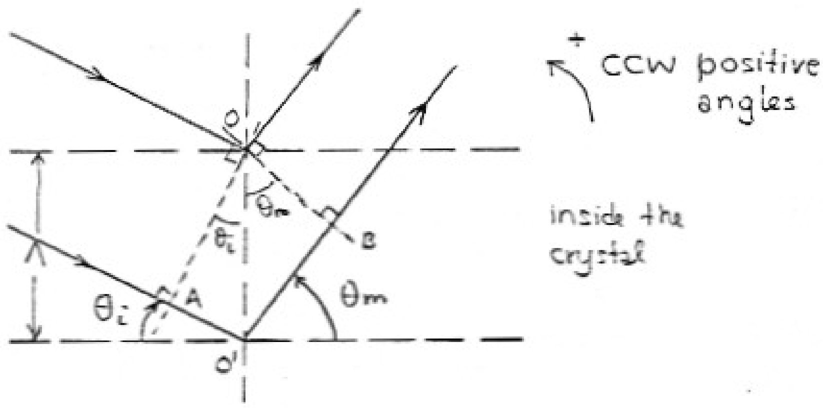
$$\Delta n(z,t) = \Delta n \sin[\omega_s t - \vec{k}_s \cdot \vec{r}]$$

$$v_s = \omega_s / k_s \qquad k_s = \frac{2\pi}{\lambda_s} = \frac{2\pi}{\Lambda}$$



This periodic change of the refractive index forms a grating. This periodic change of the refractive index (grating) can diffract an incident beam of light. The possible directions of the diffracted beams (there can be multiple diffracted beams) can be calculated using the following procedure.

Assume a period of the induced grating and two rays of the incident beam at an angle θ_i and of the diffracted beam at an angle θ_m.

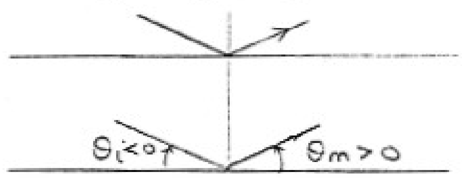


In order for the diffracted wave to build up, it is necessary that the optical path difference $AO' + O'B$ is multiple of wavelength (inside the crystal). This can be written as

$\Lambda(\sin\theta_i + \sin\theta_m) = m\,\frac{\lambda_0}{n}$ where n is the average refractive index of the crystal. If we make the convention that all CCW angles are positive (as measured from the horizontal) and all CW angles are negative, then the above equation can be written as

$$\Lambda(\sin\theta_i - \sin\theta_m) = m\,\frac{\lambda_0}{n} \qquad m = 0, \pm 1, \pm 2, \ldots$$

The previous equation is also called the grating equation. For $m = 0 \Rightarrow \sin\theta_i = \sin\theta_m \Rightarrow \theta_m = \theta_i < 0$ and it corresponds to the zero diffracted order or the transmitted beam. If $-\theta_i = \theta_m = \theta$ as shown in the following diagram,



then the above condition can be written as follows;

$$2\Lambda\sin\theta = m\,\frac{\lambda_0}{n},$$

and is called the Bragg condition.

Both the grating equation and the Bragg condition are phase conditions and they do not guarantee efficient diffraction.

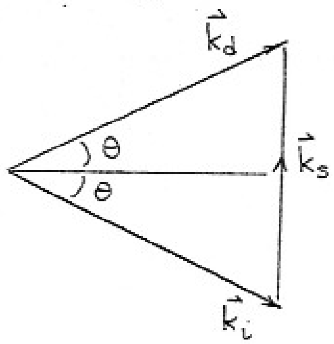
Particle Picture of Bragg Diffraction of Light by Sound

Using the wave-particle duality we can consider the optical field as photons with momentum $\hbar\vec{k}_i$ and energy $\hbar\omega_i$. Similarly, the acoustic wave can be considered as phonons of momentum $\hbar\vec{k}_s$ and energy $\hbar\omega_s$. If the diffracted optical field is comprised of photons of momentum $\hbar\vec{k}_d$ and energy $\hbar\omega_d$ then from conservation of energy and momentum we have:

$$\vec{k}_d = \vec{k}_i + \vec{k}_s \qquad \text{(momentum conservation)}$$

$$\omega_d = \omega_i + \omega_s \qquad \text{(energy conservation)}$$

Usually $\omega_s \ll \omega_i \rightsquigarrow \omega_d \simeq \omega_i$ and $|\vec{k}_i| = |\vec{k}_d|$. Graphically, this can be represented by the following diagram:



From the diagram if $k_d = k_i = k$

$$k_s = 2k\sin\theta \Rightarrow$$

$$\frac{2\pi}{\Lambda} = 2\,\frac{2\pi}{\lambda_0/n}\cdot\sin\theta \Rightarrow$$

$2\Lambda\sin\theta = \lambda_0/n$ which is the Bragg condition for the first $(m=1)$ diffracted order. In general we can have multiple diffracted waves with wavevectors and frequencies given by

$$\vec{k}_m = \vec{k}_i + m\vec{k}_s \qquad m = 0, \pm1, \pm2, \ldots$$

$$\omega_m = \omega_i + m\omega_s$$

The above generalized momentum condition is also called the Flocquet condition. For the negative orders $\omega_m = \omega_i - |m|\omega_s < \omega_i$. The resulting ω_m's correspond to Doppler shifts.

<u>Analysis of Bragg Diffraction of Light by Acoustic Waves.</u>

Let's start from the Maxwell's equations:

$$\vec{\nabla}\times\vec{e} = -\mu_o \frac{\partial \vec{h}}{\partial t} \qquad \vec{\nabla}\cdot\epsilon\vec{e} = 0 \sim \vec{\nabla}\cdot\vec{e} \simeq 0$$

$$\vec{\nabla}\times\vec{h} = \frac{\partial}{\partial t}(\epsilon_o\vec{e} + \vec{p}) \qquad \vec{\nabla}\cdot\vec{h} = 0$$

Now let's decompose $\vec{p} = \vec{p}_o + \Delta\vec{p}$, where $\vec{p}_o$ is the induced polarization without any acoustic wave and $\Delta\vec{p}$ is the change of the polarization due to the acoustic wave. The $\Delta\vec{p}$ will be treated as a perturbation. The relation between $\Delta\vec{p}$ and the index change Δn due to the acoustic wave can be found as follows:

$$\epsilon\vec{e} = \epsilon_o\vec{e} + \vec{p} \Rightarrow \vec{p} = \epsilon_o(n^2-1)\vec{e} \Rightarrow$$

$$\Delta\vec{p} = \epsilon_o 2n\Delta n\,\vec{e} = 2\epsilon_o\sqrt{\epsilon_r}\,\Delta n\,\vec{e} = 2.\sqrt{\epsilon_o\epsilon}\,\Delta n\,\vec{e}$$

and $\Delta n = \Delta n(\vec{r},t) = \Delta n_o \cos[\omega_s t - \vec{k}_s\cdot\vec{r}]$

From Maxwell's equations we get:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{e}) = -\mu_o\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}(\epsilon_o\vec{e}+\vec{p}+\Delta\vec{p})\right) = -\mu_o\frac{\partial^2}{\partial t^2}(\epsilon\vec{e}+\Delta\vec{p}) \Rightarrow$$

$$\Rightarrow \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{e}}}_{0}) - \vec{\nabla}^2\vec{e} = -\mu_o\epsilon\frac{\partial^2}{\partial t^2}\vec{e} - \mu_o\frac{\partial^2}{\partial t^2}\Delta\vec{p} \Rightarrow$$

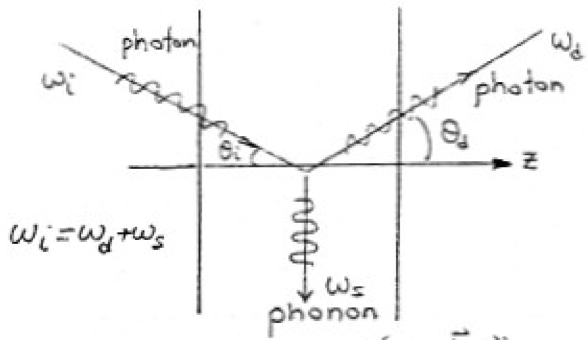
$$\vec{\nabla}^2\vec{e} - \mu_o\epsilon\frac{\partial^2\vec{e}}{\partial t^2} = \mu_o\frac{\partial^2}{\partial t^2}\Delta\vec{p}$$

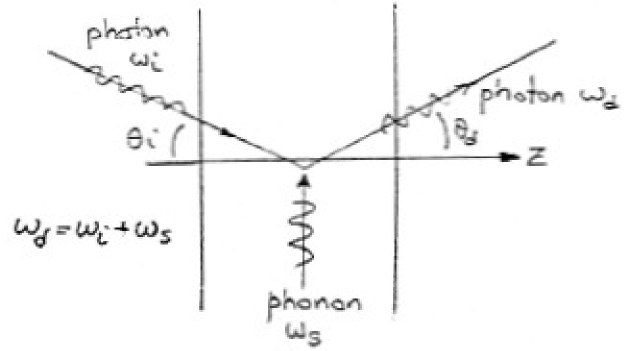
The total electric field $\vec{e}$ is the sum of the incident and the diffracted fields.

$$\vec{e} = e\,\hat{e} = \left\{\frac{1}{2}\left(A_i(z)\,e^{+j(\omega_i t - \vec{k}_i\cdot\vec{r})} + c.c.\right) + \frac{1}{2}\left(A_d(z)\,e^{+j(\omega_d t - \vec{k}_d\cdot\vec{r})} + c.c.\right)\right\}\hat{e}$$

where c.c. denotes complex conjugate and $A_i(z)$, $A_d(z)$ are slowly varying amplitudes. ω_i, ω_d are the angular frequencies of the incident and diffracted waves and $\vec{k}_i$, $\vec{k}_d$ their corresponding wavevectors. $\hat{e}$ is the polarization unit vector that remains constant.

The interaction geometry looks like:





If $e = \frac{1}{2}\left(A(z)\, e^{j(\omega t - \vec{k}\cdot\vec{r})} + c.c.\right)$ it is straightforward to show that

$$\nabla^2 e = -\frac{1}{2}\, e^{j(\omega t - \vec{k}\cdot\vec{r})} \left[\vec{k}\cdot\vec{k}A + j2k_z\frac{dA}{dz} + \frac{d^2A}{dz^2}\right] + c.c.$$

$$\mu_0\epsilon\frac{\partial^2}{\partial t^2}e = -\omega^2\mu_0\epsilon\ \frac{1}{2}A(z)\, e^{j(\omega t - \vec{k}\cdot\vec{r})} + c.c.$$

Therefore,

$$\nabla^2 e - \mu_0\epsilon\frac{\partial^2 e}{\partial t^2} = -\frac{1}{2}\left[\vec{k}_i\cdot\vec{k}_i A_i + 2jk_{iz}\frac{dA_i}{dz} + \frac{d^2A_i}{dz^2} - \mu_0\epsilon\omega_i^2 A_i\right] e^{j(\omega_i t - \vec{k}_i\cdot\vec{r})} + c.c.$$
$$-\frac{1}{2}\left[\vec{k}_d\cdot\vec{k}_d A_d + 2jk_{dz}\frac{dA_d}{dz} + \frac{d^2A_d}{dz^2} - \mu_0\epsilon\omega_d^2 A_d\right] e^{j(\omega_d t - \vec{k}_d\cdot\vec{r})} + c.c.$$

But $\vec{k}_i\cdot\vec{k}_i = \omega_i^2\mu_0\epsilon$ and $\vec{k}_d\cdot\vec{k}_d = \omega_d^2\mu_0\epsilon$ (for plane waves) and since A_i, A_d are slowly varying functions of z $\frac{d^2A_i}{dz^2} \simeq 0$, $\frac{d^2A_d}{dz^2} \simeq 0$ as compared to the $\frac{dA_i}{dz}k_{iz}$ and $\frac{dA_d}{dz}k_{dz}$ terms. Consequently,

$$\nabla^2 e - \mu_0\epsilon\frac{\partial^2 e}{\partial t^2} \simeq -\frac{1}{2}\left[+2j\,k_{iz}\frac{dA_i}{dz}\right] e^{j(\omega_i t - \vec{k}_i\cdot\vec{r})} + c.c.$$
$$-\frac{1}{2}\left[+2j\,k_{dz}\frac{dA_d}{dz}\right] e^{j(\omega_d t - \vec{k}_d\cdot\vec{r})} + c.c.$$

Now let's examine the $\mu_0\frac{\partial^2}{\partial t^2}(\Delta p)$ term:

$$\mu_0\frac{\partial^2}{\partial t^2}(\Delta p) = \mu_0\frac{\partial^2}{\partial t^2}\left[2\sqrt{\epsilon_0\epsilon}\,\Delta n\left(\frac{1}{2}A_i e^{j(\omega_i t - \vec{k}_i\cdot\vec{r})} + c.c. + \frac{1}{2}A_d e^{j(\omega_d t - \vec{k}_d\cdot\vec{r})} + c.c.\right)\right]$$

$$= \mu_0 2\sqrt{\epsilon_0\epsilon}\,\Delta n_0\frac{\partial^2}{\partial t^2}\left[\left(\frac{1}{2}e^{j(\omega_s t - \vec{k}_s\cdot\vec{r})} + c.c.\right)\left(\frac{1}{2}A_i e^{j(\omega_i t - \vec{k}_i\cdot\vec{r})} + \frac{1}{2}A_d e^{j(\omega_d t - \vec{k}_d\cdot\vec{r})} + c.c.\right)\right] =$$

$$= \mu_0 2\sqrt{\epsilon_0\epsilon}\,\Delta n_0\frac{\partial^2}{\partial t^2}\left[\frac{1}{4}A_i e^{j[(\omega_s + \omega_i)t - (\vec{k}_s + \vec{k}_i)\cdot\vec{r}]}\right] +$$

$$\frac{1}{4} A_i^* e^{j[(\omega_s-\omega_i)t-(\vec{k}_s-\vec{k}_i)\cdot\vec{r}]} +$$
$$\frac{1}{4} A_d e^{j[(\omega_s+\omega_d)t-(\vec{k}_s+\vec{k}_d)\vec{r}]} +$$
$$\frac{1}{4} A_d^* e^{j[(\omega_s-\omega_d)t-(\vec{k}_s-\vec{k}_d)\cdot\vec{r}]} +$$
$$\frac{1}{4} A_i e^{j[(-\omega_s+\omega_i)t-(-\vec{k}_s+\vec{k}_i)\cdot\vec{r}]} +$$
$$\frac{1}{4} A_i^* e^{j[-(\omega_s+\omega_i)t+(\vec{k}_s+\vec{k}_i)\cdot\vec{r}]} +$$
$$\frac{1}{4} A_d e^{j[(-\omega_s+\omega_d)t-(-\vec{k}_s+\vec{k}_d)\cdot\vec{r}]} +$$
$$\frac{1}{4} A_d^* e^{j[-(\omega_s+\omega_d)t+(\vec{k}_s+\vec{k}_d)\cdot\vec{r}]} \Bigg]$$

Now we have to keep only the in-phase terms because only these are going to contribute at steady state. Assume that

$$\omega_d = \omega_i + \omega_s \rightsquigarrow \omega_i = \omega_d - \omega_s$$

Keeping only the in-phase terms we get:

$$-\frac{1}{2}\left[+2j\,k_{iz}\frac{dA_i}{dz}\right]e^{j(\omega_i t-\vec{k}_i\cdot\vec{r})} + \text{c.c.} - \frac{1}{2}\left[+2j\,k_{dz}\frac{dA_d}{dz}\right]e^{j(\omega_d t-\vec{k}_d\cdot\vec{r})} + \text{c.c.} =$$
$$= \mu_0\, 2\sqrt{\epsilon_0\epsilon}\,\Delta n_0 \Bigg\{\left[\frac{1}{4}A_d\, e^{j[\overbrace{(\omega_d-\omega_s)}^{\omega_i}t-(\vec{k}_d-\vec{k}_s)\cdot\vec{r}]} + \text{c.c.}\right]\left[-(\omega_d-\omega_s)^2\right] +$$
$$\left[\frac{1}{4}A_i\, e^{j[\underbrace{(\omega_i+\omega_s)}_{\omega_d}t-(\vec{k}_i+\vec{k}_s)\cdot\vec{r}]} + \text{c.c.}\right]\left[-(\omega_i+\omega_s)^2\right]\Bigg\}$$

while terms like $\omega_d+\omega_s$, $\omega_i-\omega_s$ and their conjugates have been neglected. The above equation should be satisfied for all times. Therefore,

$$j\,k_{iz}\frac{dA_i}{dz}e^{-j\vec{k}_i\cdot\vec{r}} = \frac{1}{2}\mu_0\sqrt{\epsilon_0\epsilon}\,\Delta n_0\,\overbrace{(\omega_d-\omega_s)^2}^{\omega_i^2} A_d\, e^{-j(\vec{k}_d-\vec{k}_s)\cdot\vec{r}}$$
$$j\,k_{dz}\frac{dA_d}{dz}e^{-j\vec{k}_d\cdot\vec{r}} = \frac{1}{2}\mu_0\sqrt{\epsilon_0\epsilon}\,\Delta n_0\,\overbrace{(\omega_i+\omega_s)^2}^{\omega_d^2} A_i\, e^{-j(\vec{k}_i+\vec{k}_s)\cdot\vec{r}}$$

Define
$$\eta_i = \frac{1}{2}\mu_0\sqrt{\epsilon_0\epsilon}\,\frac{\Delta n_0}{k_{iz}}\cdot\omega_i^2 =$$
$$= \frac{1}{2}\mu_0\sqrt{\epsilon_0\epsilon}\,\frac{\Delta n_0}{\omega_i\sqrt{\mu_0\epsilon}\cos\theta_i}\,\omega_i^2 = \frac{1}{2}\,\frac{1}{c}\cdot\frac{\omega_i\,\Delta n_0}{\cos\theta_i}$$

Similarly, $\eta_d = \frac{1}{2}\frac{1}{c}\frac{\omega_d\,\Delta n_0}{\cos\theta_d}$. Using these definitions the above equations become:

$$\frac{dA_i}{dz} = -j\,\eta_i\,A_d\,e^{+j[\vec{k}_i+\vec{k}_s-\vec{k}_d]\cdot\vec{r}}$$

$$\frac{dA_d}{dz} = -j\,\eta_d\,A_i\,e^{-j[\vec{k}_i+\vec{k}_s-\vec{k}_d]\cdot\vec{r}}$$

For strong interaction within the volume of the acoustic wave disturbance the exponential factors need to be such that:

$$\vec{k}_i + \vec{k}_s - \vec{k}_d = 0 \qquad \text{Bragg Condition} \, .$$

If the Bragg condition is satisfied then $\theta_i = \theta_d$ and $\omega_d \simeq \omega_i$ since $\omega_s \ll \omega_i$ and the above equations become: $\left(\eta = \eta_i = \eta_d = \frac{\Delta n_0 \omega_i}{2c\cos\theta_i}\right)$

$$\frac{dA_i}{dz} = -j\eta\,A_d$$

$$\frac{dA_d}{dz} = -j\eta\,A_i$$

Solution:

It is straightforward to show that:

$$A_d(z) = A_d(0)\cos(\eta z) - j\,A_i(0)\sin(\eta z)$$

$$A_i(z) = A_i(0)\cos(\eta z) - j\,A_d(0)\sin(\eta z)$$

and assuming that at $z=0$ there is no diffracted field the result is:

$$A_d(z) = -j\,A_i(0)\sin(\eta z),$$

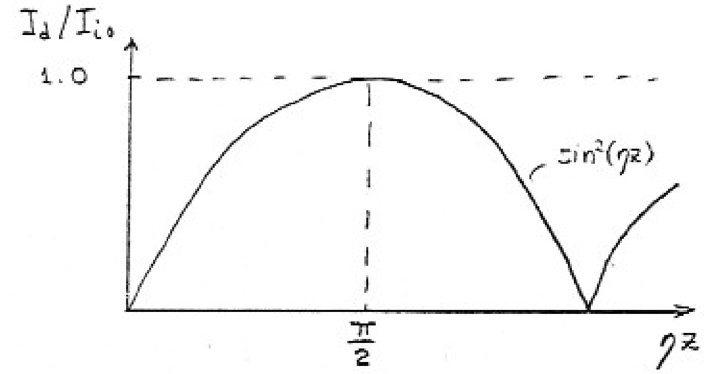
$$A_i(z) = A_i(0)\cos(\eta z).$$

Then the diffracted and transmitted power are:

$$\frac{I_d(z)}{I_i(z=0)} = \frac{|A_d(z)|^2}{|A_i(0)|^2} = \sin^2(\eta z)$$

$$\frac{I_i(z)}{I_i(z=0)} = \frac{|A_i(z)|^2}{|A_i(0)|^2} = \cos^2(\eta z)$$

which satisfy the conservation of energy since $I_d(z) + I_i(z) = I_i(0)$ for any z.



max : $\eta z_o = \pi/2$

$$\frac{I_d(z)}{I_i(0)} = \sin^2[\eta z] = \sin^2\left[\frac{\omega \Delta n_o}{2c\cos\theta_i} z\right] = \sin^2\left[\frac{\cancel{2}\pi \Delta n_o}{\cancel{2}\lambda_o \cos\theta_i} z\right] =$$

$$= \sin^2\left[\frac{\pi \Delta n_o z}{\lambda_o \cos\theta_i}\right]$$

Now let's relate Δn_o with the acoustic wave intensity.

Δn_o is related to the strain via

$$\Delta n_o = -\frac{n^3 p}{2} S_o$$

where p the photoelastic constant of the medium and S_o is the amplitude of the induced strain. S_o is related to I_{acoust} (= acoustic wave intensity in W/m^2) by

$$S_o = \sqrt{\frac{2 I_{acoust}}{\rho v_s^3}}$$ where ρ is the mass density (kgr/m^3) and

v_s is the velocity of sound in the medium. Then,

$$\frac{I_d}{I_i(0)} = \sin^2\left[\frac{\pi z}{\lambda_o \cos\theta_i} \cdot \sqrt{\frac{M I_{acoust}}{2}}\right]$$

where $M = \frac{n^6 p^2}{\rho v_s^3}$ is the figure of merit of the acousto-optic interaction.

Deflection of Light by Sound:

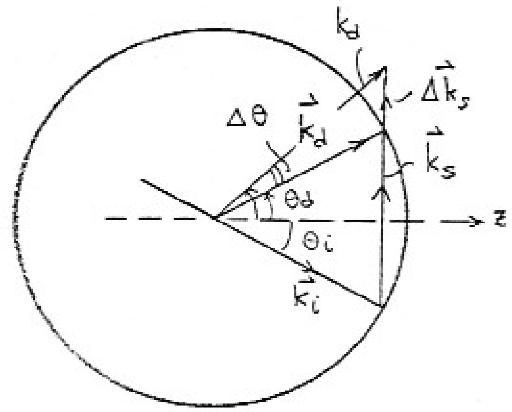
One advantage of the acousto-optic devices over the electro-optic devices is that by changing the frequency of the sound wave we can change the direction of the diffracted optical beam. This can be easily understood if we observe the grating equation:

$$\Lambda\,[\sin\theta_i + \sin\theta_m] = m\,\frac{\lambda_o}{n} \Rightarrow$$

$$\sin\theta_m = -\sin\theta_i + m\,\frac{\lambda_o/n}{\Lambda} = -\sin\theta_i + m\,\frac{\lambda_o/n}{v_s}\nu_s$$

where $\Lambda = \lambda_s = v_s/\nu_s$ and ν_s is the sound frequency.

Now let's assume that the Bragg condition is satisfied. This is shown in the following diagram.



At the Bragg condition

$\theta_i = \theta_d = \theta$ and

$$2\Lambda\sin\theta = m\left(\frac{\lambda_o}{n}\right)$$

For $m = 1$ (1st. order)

$$2\Lambda\sin\theta = \lambda_o/n$$

By changing ν_s by $\Delta\nu_s$, $\vec{k}_s$ is also changing by $\Delta\vec{k}_s$.

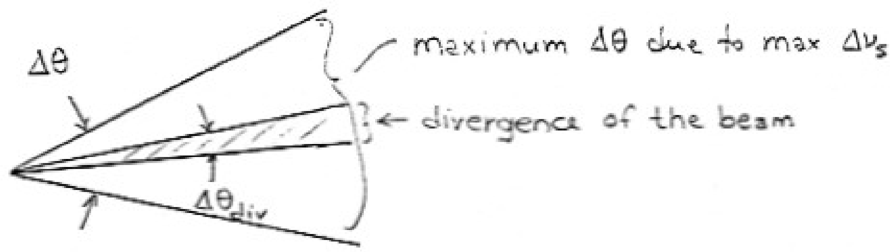
From the grating equation we have

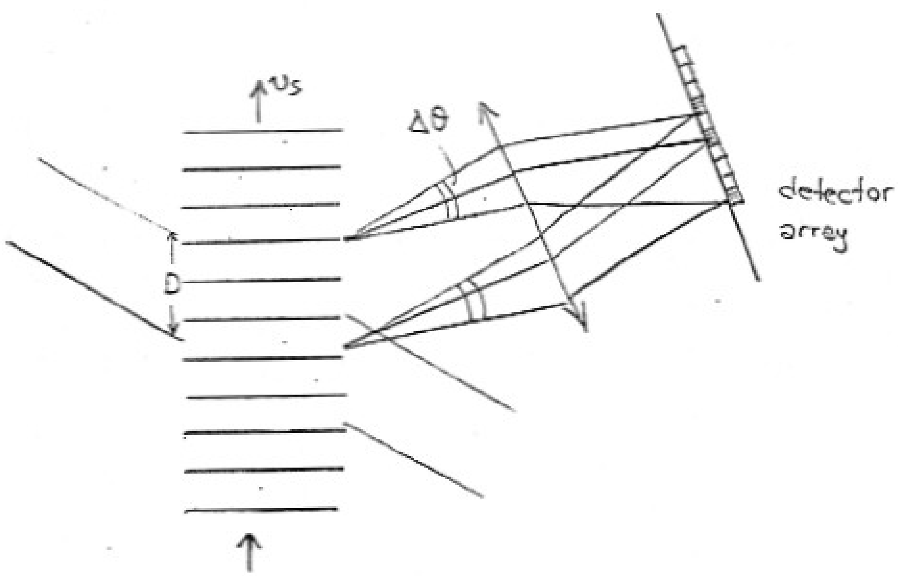
$$\sin\theta_d = -\sin\theta_i + \frac{\lambda_o/n}{v_s}\nu_s \Rightarrow \cos\theta_d\cdot\Delta\theta = \frac{\lambda_o/n}{v_s}\Delta\nu_s$$

and since θ_i, θ_d are small close to the Bragg condition

$$\Delta\theta \simeq \frac{\lambda_o/n}{v_s}\cdot\Delta\nu_s$$

The important parameter is not the magnitude of the actual $\Delta\theta$ due to a change $\Delta\nu_s$ in frequency but how this angle compares to the divergence of a beam. Recall that the TEM_{00} mode has a beam divergence of $\Delta\theta_{div} \simeq \frac{4}{\pi}\frac{\lambda_0/n}{D}$ where D is the beam diameter.





The number of resolvable spots is:

$$N = \frac{\Delta\theta}{\Delta\theta_{div}} = \frac{\frac{\lambda_0/n}{v_s}\Delta\nu_s}{\left(\frac{4}{\pi}\right)\frac{\lambda_0/n}{D}} = \left(\frac{\pi}{4}\right)\cdot\frac{D}{v_s}\cdot\Delta\nu_s \simeq T\cdot B$$

where $T = \frac{D}{v_s}$ = transit time for the sound to cross the light beam and B is the acoustic signal bandwidth.

$T \cdot B$ is also called the time bandwidth product.

Example:

Assume an acousto-optic cell on $LiNbO_3$:

$v_s = 6.57\ km/sec$, $\rho = 4.7 \cdot 10^3\ kg/m^3$, $n = 2.214$ (at $\lambda_o = 0.6328\mu m$)

$p = 0.15$, $I_{acoust} = 10^6 W/m^2$, $\nu_s = 100\ MHz$

$$M = \frac{n^6 p^2}{\rho v_s^3} = \frac{2.214^6 \cdot 0.15^2}{4.7 \cdot 10^3\ (6.57 \cdot 10^3)^3} \cdot \frac{sec^3}{kg} = 1.99 \cdot 10^{-15}\ \frac{sec^3}{kg} = 1.99 \cdot 10^{-15}\ \frac{m^2}{W}$$

$$\Delta n_o = \left(\frac{1}{2} M I_{acoust}\right)^{1/2} = 3.15 \cdot 10^{-5}$$

$$\Lambda = \lambda_s = \frac{v_s}{\nu_s} = \frac{6.57 \cdot 10^3}{10^8} = 65.7\ \mu m$$

Bragg condition:

$$\frac{\lambda_o}{n_o} = 2\Lambda \sin\theta \Rightarrow \sin\theta = \frac{\lambda_o / n_o}{2\Lambda} \Rightarrow \theta = 0.125°$$

$$\Delta\theta \simeq \frac{\lambda_o / n_o}{v_s} \cdot \Delta f_s = 0.249°$$

for $\Delta f_s = 100 MHz$