Notes on Electro-Optics

Review of Basic Principles of Electromagnetic Fields

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REVIEW OF BASIC PRINCIPLES OF ELECTROMAGNETIC FIELDS †

1. Review of Maxwell's Equations

The basis of electromagnetic fields theory are Maxwell's equations. These equations describe how the electromagnetic fields are coupled together and are related to the electric charges and currents (which are the sources of the electromagnetic fields). The most common representation of the Maxwell's equations is the following:

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \tag{1}$$

$$\vec{\nabla} \times \vec{\mathcal{H}} = \vec{J} + \frac{\partial \vec{\mathcal{D}}}{\partial t}, \tag{2}$$

$$\vec{\nabla} \cdot \vec{\mathcal{D}} = \rho, \tag{3}$$

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = 0, \tag{4}$$

where $\vec{\mathcal{E}}(\vec{r}, t)$ represents the electric field (in V/m), $\vec{\mathcal{D}}(\vec{r}, t)$ represents the displacement vector (or the electric flux density vector in C/m), $\vec{\mathcal{H}}(\vec{r}, t)$ represents the magnetic field (in A/m), $\vec{\mathcal{B}}(\vec{r}, t)$ represents the magnetic flux density (in Tesla = Wb/m^2), $\vec{\mathcal{J}}(\vec{r}, t)$ represents the electric current density (in A/m^2),and $\rho(\vec{r}, t)$ represents the electric charge density (in C/m^3). The above equations are written in the form of differential equations while an integral form can also be used. In the integral form the Maxwell's equations are written as follows:

$$\oint_C \vec{\mathcal{E}} \cdot d\vec{\ell} = -\frac{d}{dt} \iint_S \vec{\mathcal{B}} \cdot d\vec{S}, \quad \text{(Faraday Law)}, \tag{5}$$

$$\oint_C \vec{\mathcal{H}} \cdot d\vec{\ell} = \int_S \vec{\mathcal{J}} \cdot d\vec{S} + \frac{d}{dt} \iint_S \vec{\mathcal{D}} \cdot d\vec{S}, \quad \text{(Ampere Law)}, \tag{6}$$

$$\oiint_S \vec{\mathcal{D}} \cdot d\vec{S} = \iiint_V \rho dV, \quad \text{(Gauss Law)}, \tag{7}$$

$$\oiint_S \vec{\mathcal{B}} \cdot d\vec{S} = 0, \quad \text{(Absence of Magnetic Monopoles)}, \tag{8}$$

where in the first two equations C is a closed contour and S is a surface ending in that contour, while the orientation of its unit vector is compatible with the right-hand rule when the direction that the contour is traced. In the last two of the above integral equations, S is a closed surface that defines a volume V and its unit vector points away from the surface.

In space with various materials parameters, the differential form of the Maxwell's equations requires the use of boundary conditions between the electromagnetics fields, when they are used at the boundaries between various media. The boundary conditions in electromagnetics can be easily derived from the integral form of Maxwell's equations by shrinking the contours, surfaces, and volumes to points on the boundary between differing regions. Then the resulting equations, known as boundary conditions, are the following:

$$\hat{i}_n \times (\vec{\mathcal{E}}_+ - \vec{\mathcal{E}}_-)_S = 0, \tag{9}$$

$$\hat{i}_n \times (\vec{\mathcal{H}}_+ - \vec{\mathcal{H}}_-)_S = \vec{\mathcal{K}}, \tag{10}$$

$$\hat{i}_n \cdot (\vec{\mathcal{D}}_+ - \vec{\mathcal{D}}_-)_S = \sigma, \tag{11}$$

$$\hat{i}_n \cdot (\vec{\mathcal{B}}_+ - \vec{\mathcal{B}}_-)_S = 0, \tag{12}$$

where $\vec{\mathcal{K}}$ is the possible surface current density (in A/m) and σ is the possible surface charge density (in C/m^2) on the boundary, respectively. The $\hat{i}_n$ is the unit vector normal to the boundary and pointing to the + region (Fig. 1).

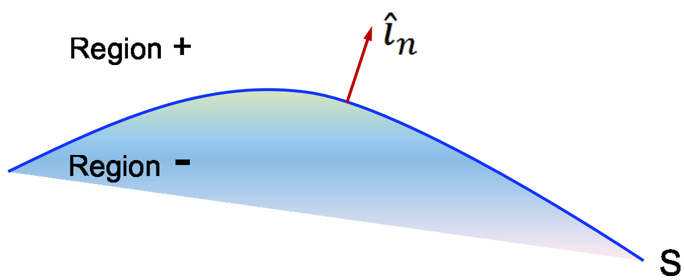


Figure 1: Boundary between two media denoted by "+" and "−" respectively.

Another equation that can be derived from Maxwell's equations is the continuity equation (conservation of electric charge) that can be written in the following form (differential or integral)

$$\vec{\nabla} \cdot \vec{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0 \Longleftrightarrow \oiint_S \vec{\mathcal{J}} \cdot d\vec{S} + \frac{d}{dt} \iiint_V \rho dV = 0, \tag{13}$$

while the corresponding boundary condition is

$$\hat{i}_n \cdot (\vec{\mathcal{J}}_+ - \vec{\mathcal{J}}_-)_S = -\vec{\nabla}_2 \cdot \vec{\mathcal{K}} - \frac{\partial \sigma}{\partial t}, \tag{14}$$

and $\vec{\nabla}_2 \cdot \vec{\mathcal{K}}$ denotes the two-dimensional divergence of the current density.

Maxwell's equations can also be written in the time-harmonic form where phasors are used. For example the real electric field, $\vec{\mathcal{E}}(\vec{r}, t)$, and its phasor representation, $\vec{E}(\vec{r}, \omega)$, are related by $\vec{\mathcal{E}}(\vec{r}, t) = \mathrm{Re}\{\vec{E}(\vec{r}, \omega) \exp(j\omega t)\}$, where ω is the radial (angular) frequency of the electromagnetic field. The same phasor representation can be used for all the electromagnetic fields and sources.

The phasor representation can be thought as a special case of the Fourier transform of the a sinusoidal time varying electromagnetic field. Using either the Fourier transform of Eqs. (1)-(4) or their phasor representation they can be written in the following time-harmonic form:

$$
\begin{aligned}
\vec{\nabla} \times \vec{E}(\vec{r},\omega) &= -j\omega \vec{B}(\vec{r},\omega), && (15)\\
\vec{\nabla} \times \vec{H}(\vec{r},\omega) &= \vec{J}(\vec{r},\omega) + j\omega \vec{D}(\vec{r},\omega), && (16)\\
\vec{\nabla} \cdot \vec{D}(\vec{r},\omega) &= \rho(\vec{r},\omega), && (17)\\
\vec{\nabla} \cdot \vec{B}(\vec{r},\omega) &= 0. && (18)
\end{aligned}
$$

The electromagnetic fields are also related via the constitutive equations. For example, in freespace, the constitutive equations are

$$
\begin{aligned}
\vec{\mathcal{D}}(\vec{r},t) = \epsilon_0 \vec{\mathcal{E}}(\vec{r},t) \iff \vec{D}(\vec{r},\omega) = \epsilon_0 \vec{E}(\vec{r},\omega), && (19)\\
\vec{\mathcal{B}}(\vec{r},t) = \mu_0 \vec{\mathcal{H}}(\vec{r},t) \iff \vec{B}(\vec{r},\omega) = \mu_0 \vec{H}(\vec{r},\omega), && (20)
\end{aligned}
$$

where $\epsilon_0 = 8.854187817 \times 10^{-12}\,\mathrm{F/m}$ is the permittivity and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$ is the permeability, respectively, of freespace. For a linear, homogeneous, isotropic medium the material response through the polarization, magnetization (and for conductive material conductivity/resistance) have to be taken into account. The relation between the material polarization and magnetization and the electromagnetic fields can be written in the form (for linear, homogeneous, and isotropic media)

$$
\begin{aligned}
\vec{\mathcal{P}}(\vec{r},t) = \epsilon_0 \int_0^\infty G_e(\tau)\vec{\mathcal{E}}(\vec{r},t-\tau)d\tau \iff \vec{P}(\vec{r},\omega) = \epsilon_0 \chi_e(\omega)\vec{E}(\vec{r},\omega), && (21)\\
\vec{\mathcal{M}}(\vec{r},t) = \int_0^\infty G_m(\tau)\vec{\mathcal{H}}(\vec{r},t-\tau)d\tau \iff \vec{M}(\vec{r},\omega) = \chi_m(\omega)\vec{H}(\vec{r},\omega), && (22)\\
\vec{\mathcal{J}}(\vec{r},t) = \int_0^\infty G_c(\tau)\vec{\mathcal{E}}(\vec{r},t-\tau)d\tau \iff \vec{J}(\vec{r},\omega) = \sigma(\omega)\vec{E}(\vec{r},\omega), && (23)
\end{aligned}
$$

where G_e, G_m, and G_c are kernels that describe the material memory of the electromagnetic fields (the intervals from 0 to ∞ take into account the causality of the material, i.e. the polarization, for example, cannot depend on future values of the electric field). The parameters χ_e, χ_m, and σ define the electric susceptibility, the magnetic susceptibility, and the conductivity of the material (these are actually the Fourier transforms of the kernels G_e, G_m, and G_c respectively). Using the above equations in the frequency domain the following constitutive equations can be written:

$$
\begin{aligned}
\vec{D}(\vec{r},\omega) &= \epsilon_0 \vec{E}(\vec{r},\omega) + \vec{P}(\vec{r},\omega) = \epsilon_0 \left[1 + \chi_e(\omega)\right] \vec{E}(\vec{r},\omega) = \epsilon(\omega)\vec{E}(\vec{r},\omega), && (24)\\
\vec{B}(\vec{r},\omega) &= \mu_0[\vec{H}(\vec{r},\omega) + \vec{M}(\vec{r},\omega)] = \mu_0 \left[1 + \chi_m(\omega)\right] \vec{H}(\vec{r},\omega) = \mu(\omega)\vec{H}(\vec{r},\omega), && (25)\\
\vec{J}(\vec{r},\omega) &= \sigma(\omega)\vec{E}(\vec{r},\omega), && (26)
\end{aligned}
$$

where $\epsilon(\omega) = \epsilon_0(1 + \chi_e) = \epsilon_0\varepsilon_r(\omega)$ is the material permittivity and ε_r is the material relative permittivity. The material permeability is $\mu(\omega) = \mu_0(1 + \chi_m) = \mu_0\mu_r(\omega)$ and μ_r is its relative permeability. The dependence of the materials macroscopic parameters (permittivity, permeability, conductivity) on the frequency denotes what is called dispersion. All real materials have dispersion. Ideally, when a material is considered as dispersion free, then its macroscopic parameters are frequency independent and Eqs. (24)-(26) are also valid in the time domain.

When a material is linear, homogeneous, and anisotropic then the constitutive equations can be written in the form

$$\vec{D}(\vec{r},\omega) = \tilde{\epsilon}(\omega)\vec{E}(\vec{r},\omega), \tag{27}$$

$$\vec{B}(\vec{r},\omega) = \tilde{\mu}(\omega)\vec{H}(\vec{r},\omega), \tag{28}$$

$$\vec{J}(\vec{r},\omega) = \tilde{\sigma}(\omega)\vec{E}(\vec{r},\omega), \tag{29}$$

where $\tilde{\epsilon}$, $\tilde{\mu}$, and $\tilde{\sigma}$ are the permittivity, permeability, and conductivity tensors (3×3 matrices in this case). Even more complex constitutive equations can describe real materials such as bianisotropic materials (where $\vec{D}$ depends on both $\vec{E}$ and $\vec{H}$ and in analogous manner $\vec{B}$ depends on both $\vec{H}$ and $\vec{E}$). Furthermore, except being anisotropic (or bianisotropic), dispersive (or non-dispersive) a material can also be inhomogeneous and/or nonlinear. In the case of the inhomogeneity the macroscopic parameters are spatially dependent (an example would be a holographic grating region). In the case of a nonlinear material the polarization, and/or magnetization and/or conductivity depend nonlinearly on the electromagnetic fields.

2. Plane Wave Solutions

2.1 Isotropic Materials

In this section it is assumed that the materials of interest are lossless dielectrics (of zero conductivity, i.e. $\sigma = 0$). In addition, it is assumed that there are no sources (electric charges, $\rho = 0$ and electric currents, $\vec{J} = 0$). This actually means that the electromagnetic fields were generated at infinite distance away from the areas of interest. Furthermore, it is assumed that all media are linear, homogeneous, isotropic, and nonmagnetic (common for optical materials). Solutions of the form (time-harmonic) $\vec{E}(\vec{r},\omega) = \vec{E}_0 \exp(-j\vec{k}\cdot\vec{r})$ are sought (phasors). The real field can be determined from $\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\{\vec{E}_0 \exp(-j\vec{k}\cdot\vec{r})\exp(j\omega t)\}$. This form of solution for the electric field constitutes a plane wave because the locus of constant phase (wavefront) is an infinite plane perpendicular to the direction of propagation that is defined through the wavevector $\vec{k}$. Using also the constitutive equations [Eqs. (24) and (25)] Maxwell's equations can be written in the following form:

$$\vec{k} \times \vec{E} = \omega\mu_0\vec{H}, \tag{30}$$

$$\vec{k} \times \vec{H} = -\omega\epsilon_0\varepsilon_r\vec{E}, \tag{31}$$

$$\vec{k} \cdot \vec{E} = 0, \tag{32}$$

$$\vec{k} \cdot \vec{H} = 0. \tag{33}$$

By eliminating the magnetic field from the above equations it is straightforward to show that the following equation should be satisfied

$$\left[\vec{k}\cdot\vec{k} - \omega^2\epsilon_0\mu_0 n^2\right]\vec{E} = 0, \tag{34}$$

where $n^2 = \varepsilon_r$ with n defined as the index of refraction of the material. Equation (34) is actually the wave equation for the case of plane wave solutions in the time harmonic form. Of course, in order for Eq. (34) to have nontrivial solutions it is necessary that the following dispersion equation be satisfied

$$\vec{k}\cdot\vec{k} - \omega^2\epsilon_0\mu_0 n^2 = \vec{k}\cdot\vec{k} - k_0^2 n^2 = 0, \tag{35}$$

where $k_0 = \omega\sqrt{\epsilon_0\mu_0} = \omega/c = 2\pi/\lambda_0$, with k_0, c, λ_0 being the freespace wavenumber, the freespace light velocity, and the freespace wavelength, respectively.

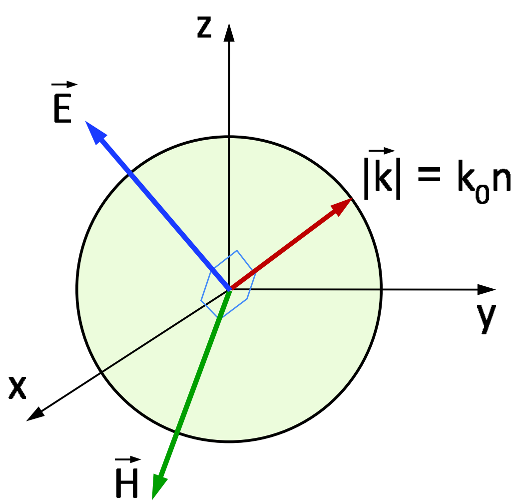


Figure 2: A plane wave dispersion sphere of radius $|\vec{k}| = k_0 n$. The electric field, the magnetic field, and the wavevector form a right-handed orthogonal system.

As it is implied from Eqs. (30)-(33) the electric field, the magnetic field, and the wavevector form a right-handed orthogonal system of vectors. Furthermore, Eq. (35) represents a sphere in wavevector space of radius $k_0 n$. I.e., for any direction of propagation of an electromagnetic wave in an isotropic, homogeneous, and linear medium the index of refraction is constant while the polarization remains perpendicular to the direction of propagation. The wavevector surface (sphere) is shown in Fig. 2. For the latter reason, usually two orthogonal polarizations are recognized which are called eigen-polarizations since they propagate inside the medium without being altered. For isotropic media the selection of the two orthogonal polarizations is arbitrary since there are infinite pairs of orthogonal polarizations for each direction of propagation. This situation changes dramatically in the case of anisotropic materials.

2.2 Anisotropic Materials

Now it is assumed that the materials of interest are lossless, homogeneous, linear, and anisotropic [1]. In other words the permittivity is a tensor, $\tilde{\epsilon} = \epsilon_0 \tilde{\varepsilon}_r$, with $\tilde{\epsilon}_{ij} = \epsilon_0 \tilde{\varepsilon}_{r,ij}$ where $i, j = x, y, z$ and $\varepsilon_{r,ij}$ are the relative permittivity tensor elements. It is reminded that when the tensor permittivity is expressed in the principal axes system then the tensor is in a diagonal form, $\tilde{\epsilon} = \epsilon_0 diag[\varepsilon_{r,xx}, \varepsilon_{r,yy}, \varepsilon_{r,zz}]$ (where "$diag$" denotes a diagonal matrix). When two of the diagonal relative permittivity elements are equal while the third is different then the material is uniaxial (one optic axis exists) while if all three differ from each other then the material is biaxial (two optic axes exist). In order to determine plane wave solutions in the case of a general anisotropic (biaxial) material the following (time-harmonic) form of the Maxwell's equations must be solved:

$$\vec{k} \times \vec{E} = \omega \mu_0 \vec{H}, \tag{36}$$

$$\vec{k} \times \vec{H} = -\omega \epsilon_0 \tilde{\varepsilon}_r \vec{E}, \tag{37}$$

$$\vec{k} \cdot \vec{D} = \epsilon_0 \vec{k} \cdot \left(\tilde{\varepsilon}_r \vec{E} \right) = 0, \tag{38}$$

$$\vec{k} \cdot \vec{H} = 0. \tag{39}$$

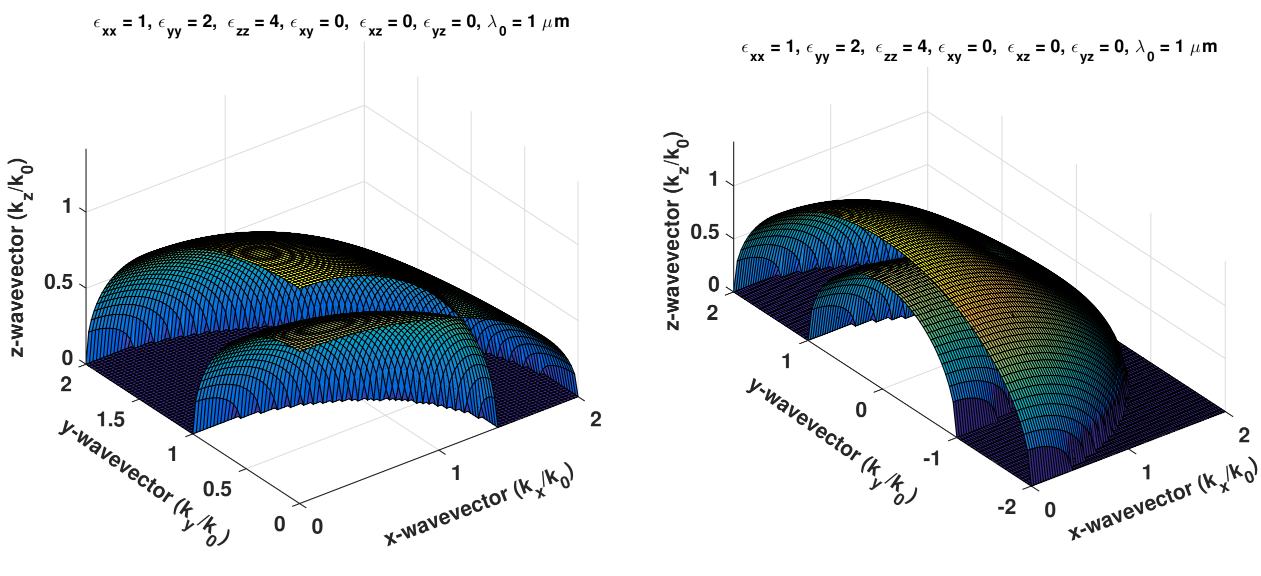
By eliminating the magnetic field it is straightforward to show that the electric field should satisfy the equation

$$\vec{k}(\vec{k} \cdot \vec{E}) - (\vec{k} \cdot \vec{k})\vec{E} = -k_0^2 \tilde{\varepsilon}_r \vec{E}. \tag{40}$$

Notice that the above equation is the wave equation for a general anisotropic material in the case of a plane wave solution. Expressing the wavevector, the electric field, and the relative permittivity tensor in the Cartesian x, y, z coordinate system the following equation must be satisfied:

$$\begin{bmatrix} k_0^2\varepsilon_{r,xx} - (k_y^2 + k_z^2) & k_x k_y + k_0^2\varepsilon_{r,xy} & k_x k_z + k_0^2\varepsilon_{r,xz} \\ k_y k_x + k_0^2\varepsilon_{r,yx} & k_0^2\varepsilon_{r,yy} - (k_x^2 + k_z^2) & k_y k_z + k_0^2\varepsilon_{r,yz} \\ k_z k_x + k_0^2\varepsilon_{r,zx} & k_z k_y + k_0^2\varepsilon_{r,zy} & k_0^2\varepsilon_{r,zz} - (k_x^2 + k_y^2) \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \Longrightarrow$$

$$\Longrightarrow \left[\tilde{\mathcal{A}}(k_x, k_y, k_z)\right] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0. \qquad (41)$$



(a) (b)

Figure 3: The wavevector surface of a biaxial material with $\varepsilon_{xx} = 1$, $\varepsilon_{yy} = 2$, $\varepsilon_{zz} = 4$, $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$, and freespace wavelength $\lambda_0 = 1.0\,\mu$m. (a) One eighth of the wavevector surface is shown. Observe how the two wavevector sheets intersect. (b) One quarter of the wavevector surface is shown. The intersection of the two-sheet surface defines one the optic axes. There is a similar intersection point in the symmetrical quarter which defines the second optic axes.

In order for the last equation to accept nontrivial solutions it is necessary to have the determinant of the matrix $\tilde{\mathcal{A}}$ equal to zero. I.e., the following should be true [2]

$$det\left[\tilde{\mathcal{A}}(k_x, k_y, k_z)\right] = det\left[k_0^2\tilde{\varepsilon}_r - k^2\tilde{I} + \vec{k}\vec{k}\right] = 0. \qquad (42)$$

The last equation defines the wavevector surface in the case of an anisotropic material and it is the analogous equation (35) that it is valid for the isotropic case. The term $\tilde{I}$ denotes a 3×3 unity matrix and the term $\vec{k}\vec{k}$ denotes a dyadic. The wavevector surface expressed by Eq. (42) is a lot more complicated than the spherical surface of the isotropic case. In fact, it is a two-sheeted surface in which the two sheets intersect. The intersection of the two sheets define the two optic axis. To have a better understanding a sample case is shown in Fig. 3. For each wavevector (k_x, k_y, k_z) that satisfies Eq. (42) the corresponding refractive index can be found as $n = \sqrt{k_x^2 + k_y^2 + k_z^2}/k_0$ while the corresponding polarization of the electric field vector can be determined from the null space of matrix $\tilde{\mathcal{A}}(k_x, k_y, k_z)$. Specifically, if the wavevector is expressed in the rectangular coordinate system, with the azimuthal and polar angles (ϕ and θ) specifying its direction, then

$$\vec{k} = k_0 n(a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) = k_0 n(\sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}), \tag{43}$$

where n is refractive index that the plane wave experiences when is propagating through the medium at the specified direction. By replacing the wavevector components from Eq.(43) into Eq. (42) the following bi-quadratic equation for n can be derived

$$\begin{aligned}
& An^4 + Bn^2 + C = 0, \qquad \text{where,} \qquad (44)\\
A &= a_x^4\varepsilon_{r,xx} + a_y^4\varepsilon_{r,yy} + a_z^4\varepsilon_{r,zz} + a_x^2a_y^2\varepsilon_{r,xx} + a_x^2a_z^2\varepsilon_{r,xx} + a_x^2a_y^2\varepsilon_{r,yy} + a_y^2a_z^2\varepsilon_{r,yy}\\
& a_x^2a_z^2\varepsilon_{r,zz} + a_y^2a_z^2\varepsilon_{r,zz} + 2a_xa_y^3\varepsilon_{r,xy} + 2a_x^3a_y\varepsilon_{r,xy} + 2a_xa_z^3\varepsilon_{r,xz} + 2a_x^3a_z\varepsilon_{r,xz}\\
& +2a_ya_z^3\varepsilon_{r,yz} + 2a_y^3a_z\varepsilon_{r,yz} + 2a_xa_ya_z^2\varepsilon_{r,xy} + 2a_xa_y^2a_z\varepsilon_{r,xz} + 2a_x^2a_ya_z\varepsilon_{r,yz},\\
B &= -a_x^2\varepsilon_{r,xx}\varepsilon_{r,yy} + a_x^2\varepsilon_{r,xz}^2 + a_y^2\varepsilon_{r,xy}^2 + a_y^2\varepsilon_{r,yz}^2 + a_z^2\varepsilon_{r,xz}^2 + a_z^2\varepsilon_{r,yz}^2 + a_x^2\varepsilon_{r,xy}^2 - a_y^2\varepsilon_{r,xx}\varepsilon_{r,yy}\\
& -a_x^2\varepsilon_{r,xx}\varepsilon_{r,zz} - a_z^2\varepsilon_{r,xx}\varepsilon_{r,zz} - a_y^2\varepsilon_{r,yy}\varepsilon_{r,zz} - a_z^2\varepsilon_{r,yy}\varepsilon_{r,zz} + 2a_xa_y\varepsilon_{r,xz}\varepsilon_{r,yz} - 2a_xa_y\varepsilon_{r,xy}\varepsilon_{r,zz}\\
& +2a_xa_z\varepsilon_{r,xy}\varepsilon_{r,yz} - 2a_xa_z\varepsilon_{r,xz}\varepsilon_{r,yy} + 2a_ya_z\varepsilon_{r,xy}\varepsilon_{r,xz} - 2a_ya_z\varepsilon_{r,xx}\varepsilon_{r,yz},\\
C &= \varepsilon_{r,xx}\varepsilon_{r,yy}\varepsilon_{r,zz} - \varepsilon_{r,zz}\varepsilon_{r,xy}^2 + 2\varepsilon_{r,xy}\varepsilon_{r,xz}\varepsilon_{r,yz} - \varepsilon_{r,yy}\varepsilon_{r,xz}^2 - \varepsilon_{r,xx}\varepsilon_{r,yz}^2.
\end{aligned}$$

Equation (44) always has two real positive solutions that correspond to the two extraordidary waves that can propagate for each specified direction. The polarization of the corresponding electric field can be found from Eq. (41). It is reminded that the $\vec{D}$ eigenvectors are perpendicular to each other. The previous equation can be simplified if the relative permittivity is expressed in the principal axis system and all the off-diagonal elements become zero ($\varepsilon_{r,xy} = \varepsilon_{r,xz} = \varepsilon_{r,yz} = 0$). In the latter case Eq. (44) can be written

$$\begin{aligned}
a_x^2n^2(n^2 - \varepsilon_{r,yy})(n^2 - \varepsilon_{r,zz}) &+ a_y^2n^2(n^2 - \varepsilon_{r,xx})(n^2 - \varepsilon_{r,zz}) + a_z^2n^2(n^2 - \varepsilon_{r,xx})(n^2 - \varepsilon_{r,yy})\\
&= (n^2 - \varepsilon_{r,xx})(n^2 - \varepsilon_{r,yy})(n^2 - \varepsilon_{r,zz}),
\end{aligned} \tag{45}$$

and the last equation can be expressed in the most common compact form [1]

$$\frac{a_x^2}{n^2 - \varepsilon_{r,xx}} + \frac{a_y^2}{n^2 - \varepsilon_{r,yy}} + \frac{a_z^2}{n^2 - \varepsilon_{r,zz}} = \frac{1}{n^2}, \tag{46}$$

though the latter equation cannot be used along the principal axes of the coordinate system.

In the case of a uniaxial material the permittivity tensor can be written (in the principal axis system) as $\tilde{\epsilon} = \epsilon_0 diag[\varepsilon_O, \varepsilon_O, \varepsilon_E] = \epsilon_0 diag[n_O^2, n_O^2, n_E^2]$ where n_O and n_E are the ordinary and the principal extraordinary refractive index respectively. The direction of the unique optic axis is denoted by the unit vector $\hat{c} = c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$. By decomposing all vectors in Eqs. (36)-(39) into a component along the optic axis and one transverse to the optic axis (for example for electric field E_c, and $\vec{E}_t$ respectively), they can be written in the following form [3]

$$(\vec{k}_t \times \hat{c})E_c + k_c(\hat{c} \times \vec{E}_t) = \omega\mu_0\vec{H}_t, \tag{47}$$

$$\vec{k}_t \times \vec{E}_t = \omega\mu_0 H_c\hat{c}, \tag{48}$$

$$(\vec{k}_t \times \hat{c})H_c + k_c(\hat{c} \times \vec{H}_t) = -\omega\epsilon_0 n_O^2\vec{E}_t, \tag{49}$$

$$\vec{k}_t \times \vec{H}_t = -\omega\epsilon_0 n_E^2 E_c\hat{c}, \tag{50}$$

$$n_O^2\vec{k}_t \cdot \vec{E}_t + n_E^2 k_c E_c = 0, \tag{51}$$

$$\vec{k}_t \cdot \vec{H}_t + k_c H_c = 0. \tag{52}$$

Manipulating the above equations it is straightforward to derive the wavevector surface equation equivalent to Eq. (42) for the uniaxial material. This is written as follows

$$[\vec{k} \cdot \vec{k} - k_0^2 n_O^2][n_O^2\vec{k} \cdot \vec{k} + (n_E^2 - n_O^2)(\vec{k} \cdot \hat{c})^2 - k_0^2 n_O^2 n_E^2] = 0. \tag{53}$$

From Eq. (53) the two possible solutions are evident while the corresponding polarizations are given as follows

$$\vec{k} \cdot \vec{k} - k_0^2 n_O^2 = 0 \implies \vec{E} \cdot \hat{c} = 0, \qquad \text{ordinary wave}, \tag{54}$$

$$n_O^2\vec{k} \cdot \vec{k} + (n_E^2 - n_O^2)(\vec{k} \cdot \hat{c})^2 - k_0^2 n_O^2 n_E^2 = 0, \implies \vec{H} \cdot \hat{c} = 0 \qquad \text{extraordinary wave}. \tag{55}$$

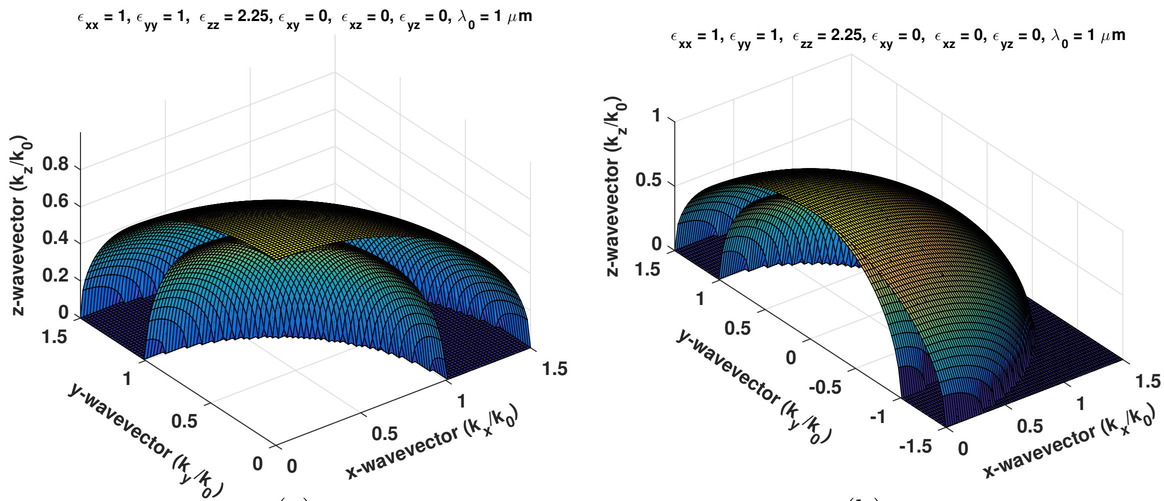
To have a better understanding a sample case is shown in Fig. 4 which is similar to Fig. 3 but for a uniaxial material.

Half of the wavevector surfaces along with their optic axes are shown in Fig. 5a and 5b for the biaxial and uniaxial cases presented previously.

For comparison purposes similar figures to Figs. 3 and 4 are given in Fig. 6 for an isotropic material of unity relative permittivity.

2.3 The Index Ellipsoid

The index ellipsoid is used to determine the eigen-polarizations and the corresponding indices of refraction for a given direction of propagation in an anisotropic crystal. The index ellipsoid (or optical indicatrix) represents a normalized surface of constant electromagnetic energy density. It is known (it is reviewed in a later section) that in a homogeneous, lossless, and linear medium (dielectric in this case) the electromagnetic energy density is given by $\langle w_{em}\rangle = (1/4)\mathrm{Re}\{\vec{E} \cdot \vec{D}^*\} = (1/4)\vec{E} \cdot \vec{D}^*$ (where $\vec{E}$ and $\vec{D}$ are the phasors of the electric field and the dielectric displacement, respectively). Replacing $\vec{E} = \epsilon_0^{-1}[\varepsilon_r]^{-1}\vec{D}$ the electromagnetic energy density can be written as (expressed in the principal axes system)



(a) (b)

Figure 4: The wavevector surface of a uniaxial material with $\varepsilon_{xx} = n_O^2 = 1$, $\varepsilon_{yy} = n_O^2 = 1$, $\varepsilon_{zz} = n_E^2 = 2.25$, $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$, and freespace wavelength $\lambda_0 = 1.0\,\mu$m. (a) One eighth of the wavevector surface is shown. The one surface is a sphere [Eq. (54)] while the other is an ellipsoid [Eq. (55)]. Observe how the two wavevector sheets become tangent in this case. (b) One quarter of the wavevector surface is shown. The two-sheet surfaces become tangent in the direction of the optic axis which in this case is the z-axis

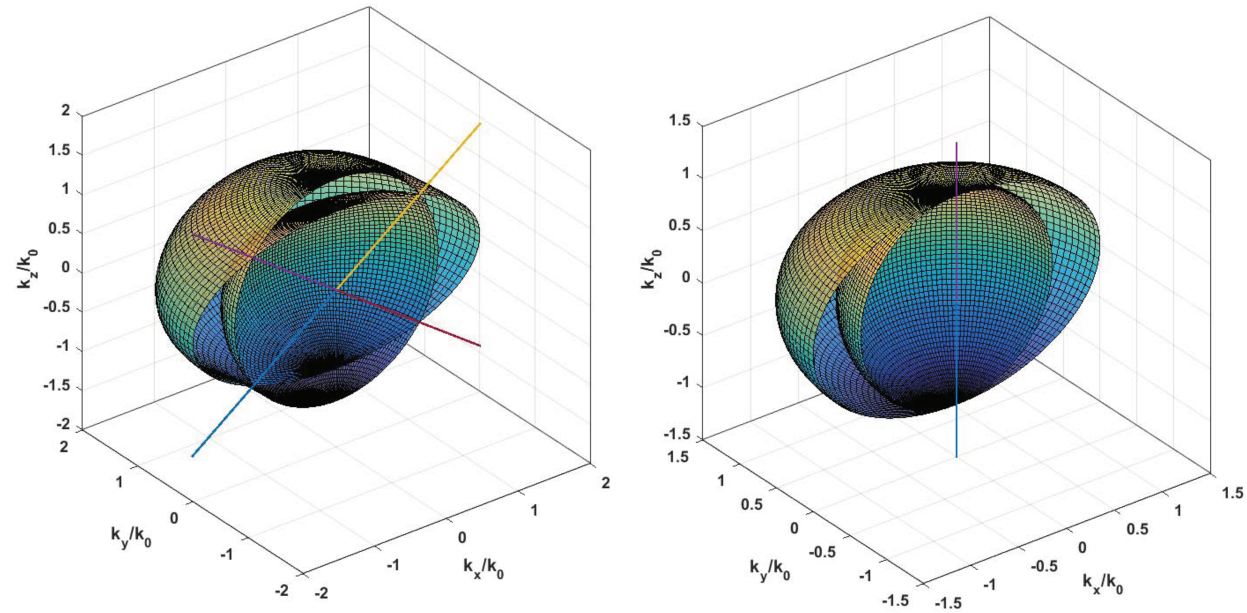
$$4\langle w_{em}\rangle\epsilon_0 = \frac{D_x^2}{\varepsilon_{r,xx}} + \frac{D_x^2}{\varepsilon_{r,yy}} + \frac{D_z^2}{\varepsilon_{r,zz}} = \frac{D_x^2}{n_{xx}^2} + \frac{D_y^2}{n_{yy}^2} + \frac{D_z^2}{n_{zz}^2}. \tag{56}$$

If in the above equation the term $\vec{D}/\sqrt{4\langle w_{em}\rangle\epsilon_0}$ is replaced by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ the resulting equation is

$$\frac{x^2}{n_{xx}^2} + \frac{y^2}{n_{yy}^2} + \frac{z^2}{n_{zz}^2} = 1, \tag{57}$$

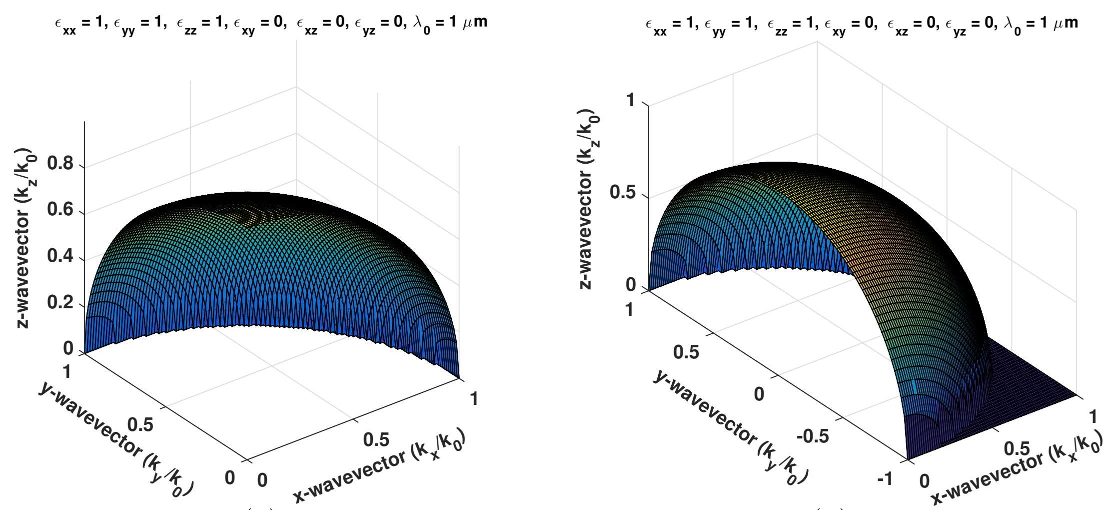
and represents the index ellipsoid in the xyz coordinate system which is shown in Fig. 7. The index ellipsoid can be used mainly to determine the two indices of refraction and the two corresponding directions of $\vec{D}$ associated with the two independent plane waves that can propagate along an arbitrary direction $\vec{k}$ in a crystal (as shown in Fig. 7). This is done as follows: the intersection ellipse between a plane through the origin that is normal to the direction of propagation $\vec{k}$ and the index ellipsoid is determined. Then the two axes of the intersection ellipse are equal in length to $2n_1$, and $2n_2$, where n_1, and n_2 are the two indices of refraction. The axes of the intersection ellipse are parallel, respectively, to the directions of the eigen-polarization vectors $\vec{D}_1$, and $\vec{D}_2$ of the two allowed solutions of the Maxwell equations [Eqs. (36)-(39)]. These indices and the corresponding eigen-polarization directions are also referred as the "slow" ($max\{n_1, n_2\}$) and the "fast" ($min\{n_1, n_2\}$). The index ellipsoid with an intersection ellipse is shown in Fig. 7 and a proof of this can be found in Ref. [1].

If the index ellipsoid is expressed in a generalized coordinate system (not the principal axes system) then it will have the form of a generalized ellipsoid in the xyz coordinate system and



(a) (b)

Figure 5: (a) The wavevector surface of a biaxial material with $\varepsilon_{xx} = 1$, $\varepsilon_{yy} = 2$, $\varepsilon_{zz} = 4$, $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$, and freespace wavelength $\lambda_0 = 1.0\,\mu$m. Half of the wavevector surface is shown. Observe the two wavevector sheets that intersect in four points that define the two optic axes of the biaxial material. (b) The wavevector surface of a uniaxial material with $\varepsilon_{xx} = 1$, $\varepsilon_{yy} = 1$, $\varepsilon_{zz} = 2.25$, $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$, and freespace wavelength $\lambda_0 = 1.0\,\mu$m. Half of the wavevector surface is shown. Observe the ellipsoidal and spherical wavevector sheets that touch only in two points where the single optic axis is defined.



(a) (b)

Figure 6: The wavevector surface of an isotropic material with $\varepsilon = n^2 = 1$, and freespace wavelength $\lambda_0 = 1.0\,\mu$m. (a) One eighth of the wavevector surface is shown. The two-sheeted surface collapses to a single spherical surface. (b) One quarter of the wavevector surface is shown.

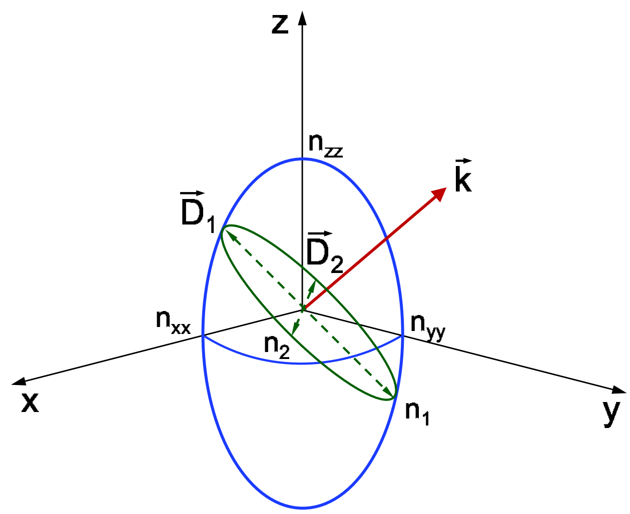


Figure 7: The index ellipsoid in the general biaxial case where $n_{xx} \neq n_{yy} \neq n_{zz} \neq n_{xx}$, shown in the principal axes system. A random direction of a plane wave wavevector is also shown. The intersection of the plane perpendicular to the wavevector forms an ellipse with its axes specifying the two eigenpolarizations $\vec{D}_1$ and $\vec{D}_2$ with n_1 and n_2 (semi-axes of the ellipse) the corresponding refractive indices.

its equation is given by

$$\begin{bmatrix} x & y & z \end{bmatrix}^T [\mathcal{A}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}^T \begin{bmatrix} \frac{1}{n_{xx}^2} & \frac{1}{n_{xy}^2} & \frac{1}{n_{xz}^2} \\ \frac{1}{n_{yx}^2} & \frac{1}{n_{yy}^2} & \frac{1}{n_{yz}^2} \\ \frac{1}{n_{zx}^2} & \frac{1}{n_{zy}^2} & \frac{1}{n_{zz}^2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1, \tag{58}$$

where $\mathcal{A} = [\varepsilon_r]^{-1}$ is the impermeability matrix. It is reminded that the impermeability matrix is also symmetric since is the inverse of the symmetric relative permittivity matrix ($n_{uv} = n_{vu}$, where $u \neq v$ and $u, v = x, y, z$).

In the case of uniaxial material where $n_{xx} = n_{yy} = n_O$ (ordinary index) and $n_{zz} = n_E$ (extraordinary index), the index ellipsoid is an ellipsoid of revolution around z axis (optic axis). In the latter case, if the direction of the electromagnetic wavevector forms an angle θ with the optic axis, the extraordinary refractive index, $n_e(\theta)$ is given by

$$\frac{\cos^2\theta}{n_O^2} + \frac{\sin^2\theta}{n_E^2} = \frac{1}{n_e^2(\theta)}, \tag{59}$$

while of course the ordinary index remains always n_O.

3. Reflection and Transmission at a Planar Boundary

In this section the reflection and transmission of a plane wave at a planar boundary between two isotropic dielectrics will be reviewed. In Fig. 8 the boundary (xz plane) between two dielectrics of permittivities $\epsilon_1 = \epsilon_0 \varepsilon_{r1} = \epsilon_0 n_1^2$ and $\epsilon_2 = \epsilon_0 \varepsilon_{r2} = \epsilon_0 n_2^2$ is shown. A plane wave is incident at angle θ_1 on the boundary and a reflected as well as a transmitted wave are induced. The electric field in region 1 (left) and region 2 (right) can be written as follows

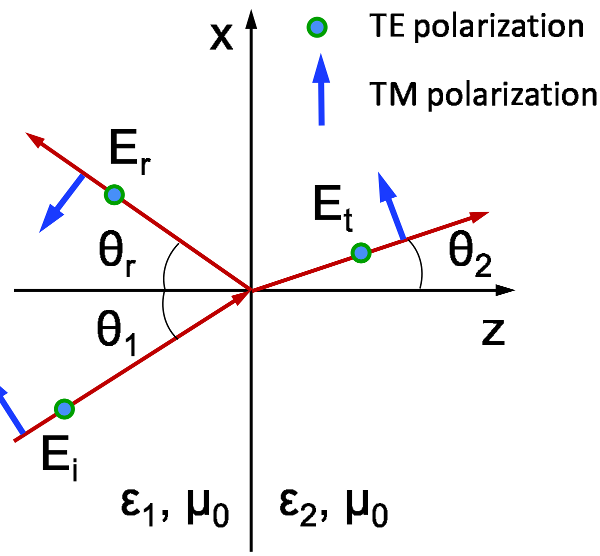


Figure 8: A planar boundary between two isotropic, nonmagnetic, dielectrics with permittivities ϵ_1 and ϵ_2. The green electric field direction (along the y-axis) corresponds to the TE polarization, while the blue electric field direction (in the xz-plane) corresponds to the TM polarization. The angle of incidence is θ_1, the angle of reflection is $\theta_r = \theta_1$, and the angle of refraction is θ_2.

$$\vec{E}_1 = \vec{E}_i \exp(-j\vec{k}_i \cdot \vec{r}) + \vec{E}_r \exp(-j\vec{k}_r \cdot \vec{r}), \tag{60}$$

$$\vec{H}_1 = \frac{1}{\omega\mu_0}\left(\vec{k}_i \times \vec{E}_i\right) \exp(-j\vec{k}_i \cdot \vec{r}) + \frac{1}{\omega\mu_0}\left(\vec{k}_r \times \vec{E}_r\right) \exp(-j\vec{k}_r \cdot \vec{r}), \tag{61}$$

$$\vec{E}_2 = \vec{E}_t \exp(-j\vec{k}_t \cdot \vec{r}), \tag{62}$$

$$\vec{H}_2 = \frac{1}{\omega\mu_0}\left(\vec{k}_t \times \vec{E}_t\right) \exp(-j\vec{k}_t \cdot \vec{r}), \tag{63}$$

where $\vec{E}_i$, $\vec{E}_r$, and $\vec{E}_t$ correspond to the incident, reflected, and transmitted electric field amplitudes, while $\vec{k}_i$, $\vec{k}_r$, and $\vec{k}_t$ correspond to the incident, reflected, and transmitted wavevectors. Of course the wavevectors satisfy Eq. (35) for each of the regions of interest. Using the continuity of the tangential to the boundary electric field components the following necessary condition must be satisfied:

$$(\vec{k}_i \cdot \vec{r})_{z=0} = (\vec{k}_r \cdot \vec{r})_{z=0} = (\vec{k}_t \cdot \vec{r})_{z=0} \implies k_{ix} = k_{rx} = k_{tx} \implies$$
$$k_0 n_1 \sin\theta_1 = k_0 n_1 \sin\theta_r = k_0 n_2 \sin\theta_2. \qquad (64)$$

The last equation is the well known *phase matching condition*. From the phase matching condition it is evident that $\theta_r = \theta_1$ (reflection angle equals incident angle) and $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (Snell's law). Of course in order to find the unknown amplitude coefficients for the reflected and the transmitted waves the full form of the boundary condition should be used in conjunction with the analogous boundary condition for the magnetic field. Usually, any incident polarization for the incident field can be decomposed into one that the electric field is perpendicular to the plane of incidence (xz plane here) which is referred as $\perp$ or TE polarization, and one that the electric field is parallel to the plane of incidence which is referred as $\|$ or TM polarization. A generalized polarization formulation is presented in a later section. These two orthogonal polarizations are decoupled in the case of the isotropic media and can be studied independently. Solving the boundary conditions the unknown amplitudes of the reflected and of the transmitted fields can be determined. These are generally expressed in the form of the Fresnel equations which are shown below:

$$r_{TE} = r_\perp = \frac{E_r}{E_i} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}, \qquad (65)$$

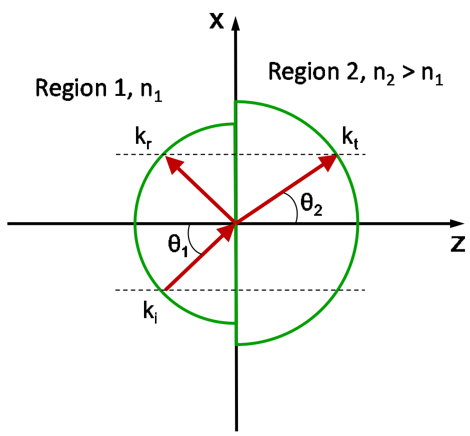
$$t_{TE} = t_\perp = \frac{E_t}{E_i} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}, \qquad (66)$$

$$r_{TM} = r_\| = \frac{E_r}{E_i} = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2}, \qquad (67)$$

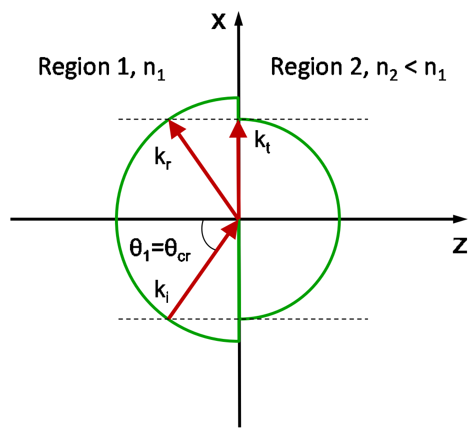
$$t_{TM} = t_\| = \frac{E_t}{E_i} = \frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2}. \qquad (68)$$

There are some interesting points to be discussed in the case of the Fresnel equations. Initially, it is assumed that $n_1 < n_2$ (for example from air to glass). In this case the wavevector diagram of Fig. 9a is valid. The phase matching condition is also shown in this figure. For this case the Brewster angle can also be defined as the angle for which the reflected wave vanishes. For nonmagnetic materials this can occur only in the case of TM ($\|$) polarization, and it is defined as $r_{TM}(\theta_1 = \theta_B) = 0 \implies \theta_B = \tan^{-1}(n_2/n_1)$. As an example, the reflection and transmission coefficients are shown in Figs. 10 and 11 as functions of the angle of incidence θ_1 for the case of $n_1 = 1.0$ and $n_2 = 1.5$.

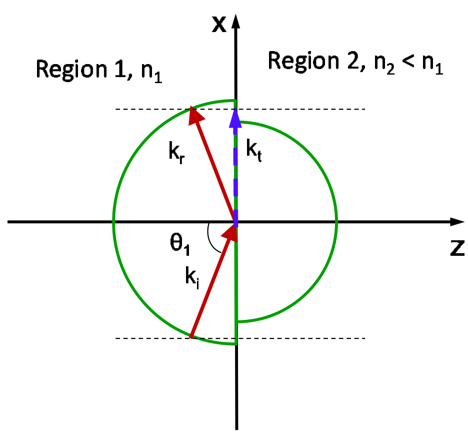
The situation becomes more interesting when $n_1 > n_2$ (for example from glass to air). In this case the Brewster angle is defined in a similar manner as before and it exists only for TM polarization. However, in this case, for both polarizations the critical angle can be defined.



(a)



(b)



(c)

Figure 9: The wavevector surface of an interface between two dielectrics. (a) Case of incidence from a low to higher refractive index ($n_1 < n_2$). (b) Case of incidence from a high to a lower refractive index ($n_1 > n_2$) and at $\theta_1 = \theta_{cr}$. (c) Case of incidence from a high to a lower refractive index ($n_1 > n_2$) and at $\theta_1 > \theta_{cr}$.

From Snell's law the critical angle is defined as the angle of incidence for which the refraction angle becomes 90 degrees. Therefore, the critical angle θ_{cr} is defined as $\theta_{cr} = \sin^{-1}(n_2/n_1)$. In wavevector space this situation is depicted in Fig. 9b. In this situation due to the phase matching condition [Eq. (64)] $k_{tx} = k_0 n_2$ and the z component of the transmitted wavevector k_{tz} becomes zero. In other words, the transmitted electric field, Eq. (62), is independent of z. This can happen only in theory since exactly at the critical angle the transmitted field is constant in region 2 which would require infinite energy from the electromagnetic field. The practical case is when the angle of incidence is greater than the critical angle ($\theta_1 > \theta_{cr}$). In the

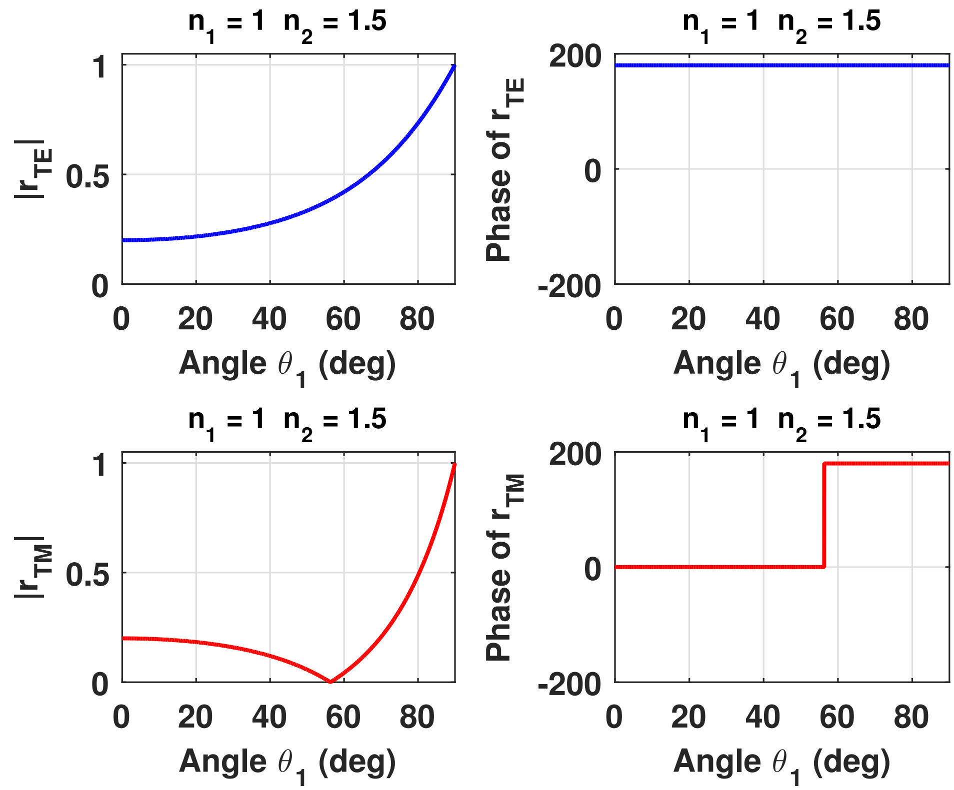


Figure 10: Reflection coefficient as a function of the angle of incidence θ_1 for TE and TM polarization in the case of $n_1 < n_2$. The Brewster angle $\theta_B = \tan^{-1}(1.5/1) = 56.31°$ is obvious in the TM polarization case.

latter case from the phase matching condition it can be deduced

$$k_{tx} = k_0 n_1 \sin\theta_1 > k_0 n_1 \sin\theta_{cr} = k_0 n_2. \tag{69}$$

From the last equation and Eq. (35) for region 2 it is obvious that (for $\theta_1 > \theta_{cr}$)

$$k_{tz}^2 = k_0^2 n_2^2 - k_{tx}^2 < 0 \Longrightarrow k_{tz} = \pm j\sqrt{k_{tx}^2 - k_0^2 n_2^2} = \pm jk_0\sqrt{n_1^2 \sin^2\theta_1 - n_2^2} = \pm j\gamma_t. \tag{70}$$

This means that the z component of the transmitted wavevector becomes purely imaginary. In the wavevector diagram this situation is depicted in Fig. 9c where the dashed purpled arrow representing the transmitted wavevector is complex. There is some ambiguity in selecting the sign of the right-hand side of Eq. (70). Since the transmitted field is expressed by Eq. (62), in order to represent a physical field the "−" sign must be selected. Then Eq. (62) becomes

$$\vec{E}_2 = \vec{E}_t \exp[-k_0\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}z]\exp[-jxk_0 n_1 \sin\theta_1] = \vec{E}_t e^{-\gamma_t z}\exp[-jxk_0 n_1 \sin\theta_1]. \tag{71}$$

The corresponding real field can be determined from the above phasor as (assuming for simplicity that $\vec{E}_i$ is real)

$$\vec{\mathcal{E}}_2(x,z,t) = |t| E_i \hat{u}_t e^{-\gamma_t z}\cos\left[\omega t - xk_0 n_1 \sin\theta_1 + \phi(\theta_1)\right], \tag{72}$$

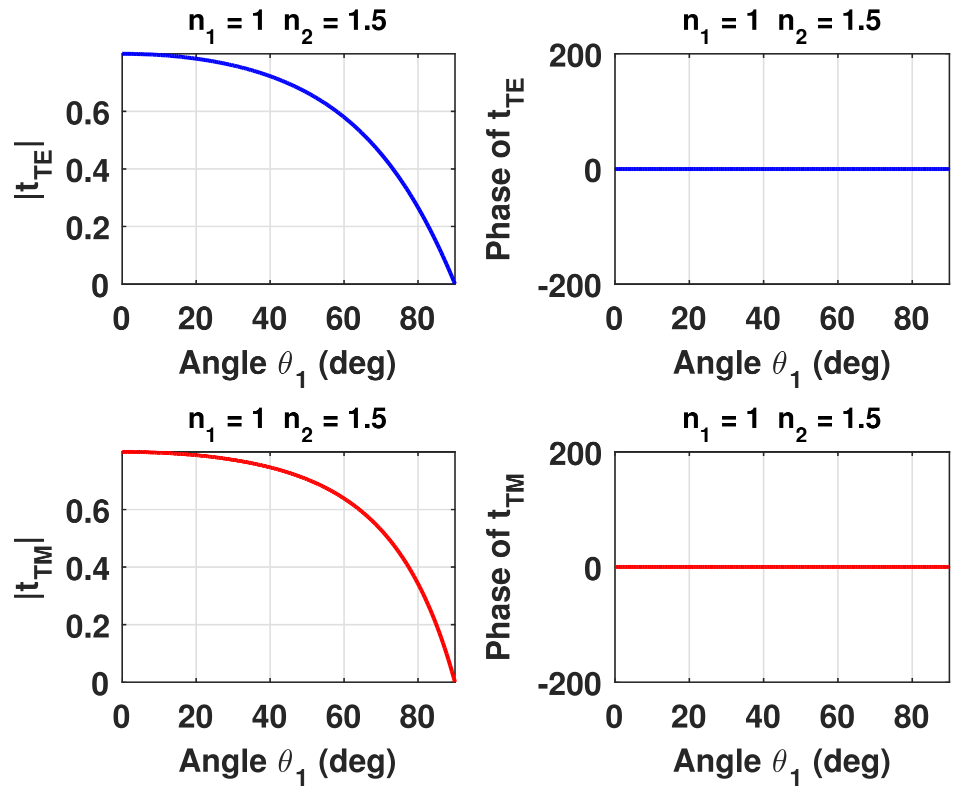


Figure 11: Transmission coefficient as a function of the angle of incidence θ_1 for TE and TM polarization in the case of $n_1 < n_2$.

where $t = |t|e^{j\phi}$ is the complex transmission coefficient and $\hat{u}_t$ is the unit vector of the transmitted electric field. For example, $\hat{u}_t = \hat{u}_{TE} = \hat{y}$ for TE polarization, and $\hat{u}_t = \hat{u}^t_{TM} = \cos\theta_2\hat{x} - \sin\theta_2\hat{z}$ for TM polarization. However, in the TM polarization case $\hat{u}^t_{TM}$ has an imaginary component that should suitably (with the appropriate phase shift) be incorporated in the latter equation of the real transmitted electric field. However, for simplicity, the form of Eq. (72) does not take into account the complex component of $\hat{u}_t$. The latter form of the transmitted field is called "evanescent field" or "evanescent wave." It is easy to show that the evanescent wave does not transfer real power in the z direction. This is the reason that this situation is called total internal reflection since all the power is reflected back into the incident region. An example evanescent field (its real part for $t = 0$ for TE polarization) is shown in Fig. 12 for $n_1 = 1.5$, $n_2 = 1.0$, and $\theta = 65°$.

In the case of total internal reflection the reflection and transmission coefficients from Fresnel equations become complex. It is interesting to define the reflection coefficients in this case which

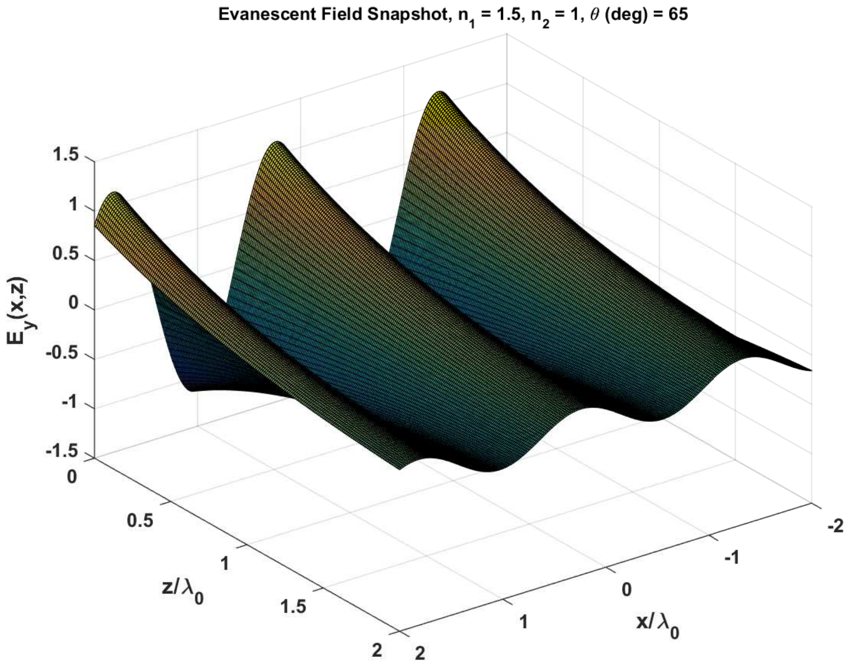


Figure 12: The real part of a TE polarized evenescent field that is generated in region 2 ($n_2 = 1.0$) from and indicent TE polarized wave from region 1 ($n_1 = 1.5$) for an angle of incidence $\theta_1 = 65°$ ($> \theta_{cr} = 41.8°$).

can be described as follows

$$r_{TE} = \frac{E_r}{E_i} = 1e^{j2\phi_{TE}(\theta_1)} = 1\exp\left[j2\tan^{-1}\left\{\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}\right\}\right], \tag{73}$$

$$r_{TM} = \frac{E_r}{E_i} = 1e^{j2\phi_{TM}(\theta_1)} = 1\exp\left[j2\tan^{-1}\left\{\frac{n_1^2}{n_2^2}\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}\right\}\right]. \tag{74}$$

From the previous equations it is evident that when $\theta_1 > \theta_{cr}$ the reflection coefficients have unit magnitude (thus all incident power is reflected back) and phase that depends on θ_1 and varies from 0 to π. The transmission coefficients become also complex in this case and are given by

$$t_{TE} = \frac{E_t}{E_i} = |t_{TE}|e^{j\phi_{TE}(\theta_1)} = \frac{2n_1\cos\theta_1}{\sqrt{n_1^2 - n_2^2}}\exp\left[j\tan^{-1}\left\{\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}\right\}\right], \tag{75}$$

$$t_{TM} = \frac{E_t}{E_i} = |t_{TM}|e^{j\phi_{TM}(\theta_1)} = \frac{2n_1 n_2\cos\theta_1}{\sqrt{n_2^4\cos^2\theta_1 + n_1^4\sin^2\theta_1 - n_1^2 n_2^2}}\exp\left[j\tan^{-1}\left\{\frac{n_1^2}{n_2^2}\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_1\cos\theta_1}\right\}\right]. \tag{76}$$

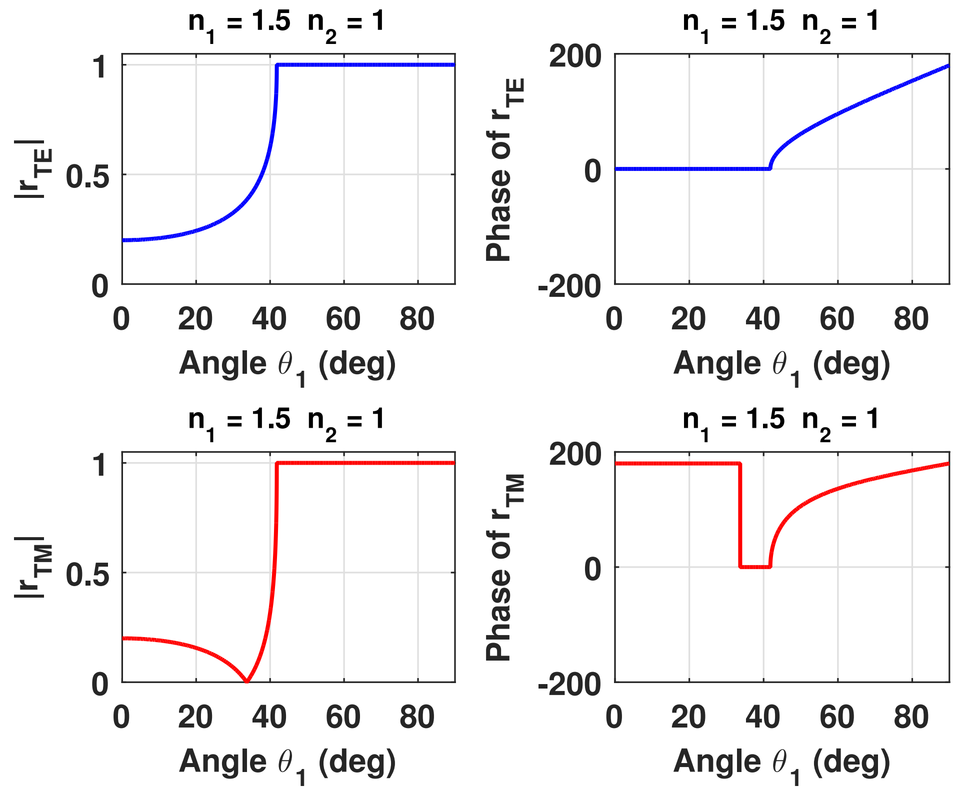


Figure 13: Reflection coefficient as a function of the angle of incidence θ_1 for TE and TM polarization in the case of $n_1 > n_2$. The Brewster angle $\theta_B = \tan^{-1}(1/1.5) = 33.69°$ is obvious in the TM polarization case. For both TE and TM polarization cases the critical angle $\theta_{cr} = \sin^{-1}(1/1.5) = 41.81°$ is also shown. For $\theta_1 > \theta_{cr}$ the reflection coefficients become complex of unity magnitude and phase $2\phi_{TE}(\theta_1)$ or $2\phi_{TM}(\theta_1)$.

As an example, the reflection and transmission coefficients are shown in Figs. 13 and 14 as functions of the angle of incidence θ_1 for the case of $n_1 = 1.5$ and $n_2 = 1.0$.

4. Electromagnetic Field Energy and Power - Poynting's Theorem

From Maxwell's equations [Eqs.(1)-(4)] it can be shown the following equality

$$
\begin{aligned}
-\vec{\nabla}\cdot(\vec{\mathcal{E}}\times\vec{\mathcal{H}}) &= \vec{\mathcal{E}}\cdot\vec{\mathcal{J}} + \vec{\mathcal{E}}\cdot\frac{\partial\vec{\mathcal{D}}}{\partial t} + \vec{\mathcal{H}}\cdot\frac{\partial\vec{\mathcal{B}}}{\partial t} = \\
&= \vec{\mathcal{E}}\cdot\vec{\mathcal{J}} + \frac{\partial}{\partial t}\Big(\frac{\epsilon_0}{2}\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}\Big) + \frac{\partial}{\partial t}\Big(\frac{\mu_0}{2}\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}\Big) + \vec{\mathcal{E}}\cdot\frac{\partial\vec{\mathcal{P}}}{\partial t} + \mu_0\vec{\mathcal{H}}\cdot\frac{\partial\vec{\mathcal{M}}}{\partial t}, \qquad (77)
\end{aligned}
$$

where the above equation represents the Poynting's theorem in point form. In case that a closed surface S is chosen, that encloses a volume V, the Poynting's theorem can also be written in

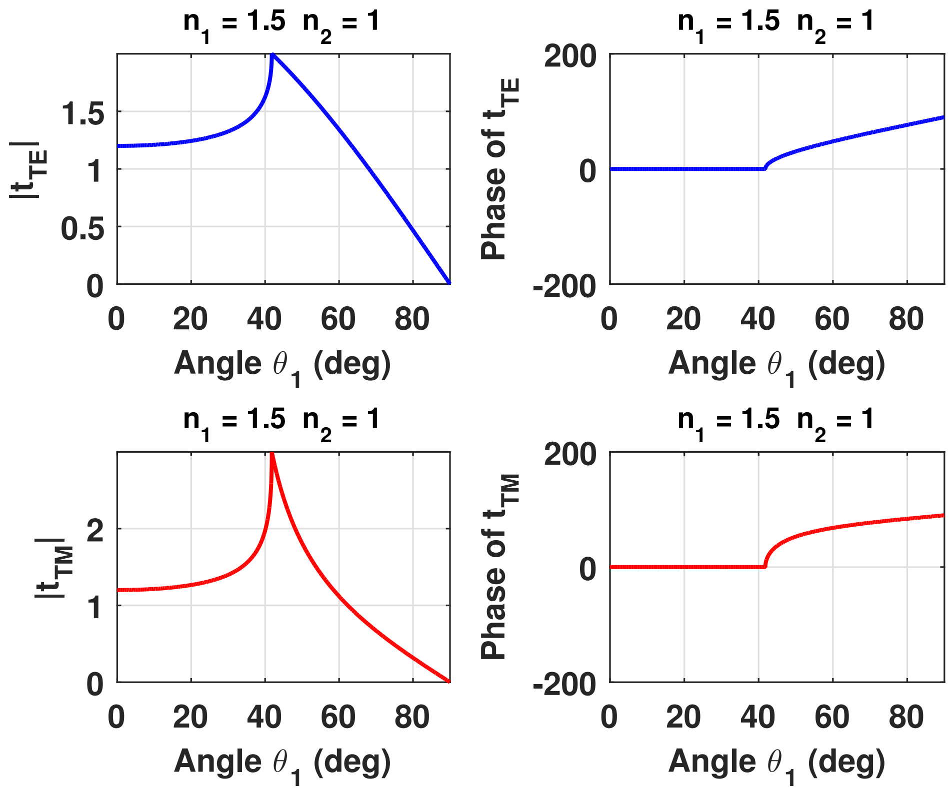


Figure 14: Transmission coefficient as a function of the angle of incidence θ_1 for TE and TM polarization in the case of $n_1 > n_2$. For $\theta_1 > \theta_{cr}$ the transmission coefficient become complex.

its integral form as follows

$$-\oiint_S \left(\vec{\mathcal{E}} \times \vec{\mathcal{H}}\right) \cdot d\vec{S} = \iiint_V \vec{\mathcal{E}} \cdot \vec{\mathcal{J}} dV + \iiint_V \frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\vec{\mathcal{E}} \cdot \vec{\mathcal{E}} + \frac{\mu_0}{2}\vec{\mathcal{H}} \cdot \vec{\mathcal{H}}\right) dV + \iiint_V \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{P}}}{\partial t} dV + \iiint_V \mu_0 \vec{\mathcal{H}} \cdot \frac{\partial \vec{\mathcal{M}}}{\partial t} dV, \tag{78}$$

where the vector $\vec{\mathcal{N}} = \vec{\mathcal{E}} \times \vec{\mathcal{H}}$ is defined as the Poynting vector (W/m^2). The left-hand side of Eq. (78) represents the total power entering the volume V via the closed surface S. The right-hand sides represent the ohmic losses expended in volume V, the rate of increase of the vacuum electromagnetic energy in volume V, the power expended in electric dipoles in volume V, and the power expended in magnetic dipoles in volume V, respectively.

If the medium is linear in terms of its electric and magnetic properties, and experiences negligible dispersion, then the electromagnetic energy density can be defined as

$$w_{em} = w_e + w_m = \frac{1}{2}\left(\vec{\mathcal{E}} \cdot \vec{\mathcal{D}} + \vec{\mathcal{H}} \cdot \vec{\mathcal{B}}\right) = \frac{1}{2}\left(\epsilon|\vec{\mathcal{E}}|^2 + \mu|\vec{\mathcal{H}}|^2\right). \tag{79}$$

Then Eqs. (77) and (78) can be written in the following forms

$$-\vec{\nabla}\cdot\left(\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right) = \vec{\mathcal{E}}\cdot\vec{\mathcal{J}}+\frac{\partial w_{em}}{\partial t}, \tag{80}$$

$$-\oiint_S\left(\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right)\cdot d\vec{S} = \iiint_V\left[\vec{\mathcal{E}}\cdot\vec{\mathcal{J}}+\frac{\partial w_{em}}{\partial t}\right]dV. \tag{81}$$

The physical meaning of the differential or integral form of Eqs. (80) or (81) is that the time rate of change of electromagnetic energy within a certain volume, plus the total work done by the fields on the sources within the volume, is equal to the energy flowing in through the boundary surfaces of the volume per unit time. This is the statement of conservation of energy.

Of course the case of a dispersionless medium is ideal. All real materials have dispersion (frequency dependent parameters) as well as losses. In order to express the Poynting theorem in the dispersive case it is necessary to express all fields in the frequency domain (Fourier transform is applied to all field quantities). For example, the real electric field in the time domain and its corresponding complex electric field in the frequency domain are related via the following Fourier transform pair

$$\vec{E}(\vec{r},\omega) = \int_{-\infty}^{+\infty}\vec{\mathcal{E}}(\vec{r},t)e^{-j\omega t}dt, \tag{82}$$

$$\vec{\mathcal{E}}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\vec{E}(\vec{r},\omega)e^{+j\omega t}d\omega, \tag{83}$$

where the same holds for all field quantities. Since the time domain fields are real it is straightforward to show that $\vec{E}(\vec{r},-\omega)=\vec{E}^*(\vec{r},\omega)$ (the "*" denotes complex conjugate) and $\epsilon^*(\omega)=\epsilon(-\omega)$ with similar arguments holding for the magnetic field counterparts. The permittivity and the permeability are in general complex functions of ω and can be written in the form

$$\epsilon(\omega) = \epsilon'(\omega)-j\epsilon''(\omega)=\epsilon_0\left[\varepsilon_r'(\omega)-j\varepsilon_r''(\omega)\right], \tag{84}$$

$$\mu(\omega) = \mu'(\omega)-j\mu''(\omega)=\mu_0\left[\mu_r'(\omega)-j\mu_r''(\omega)\right], \tag{85}$$

where the real and the imaginary parts of the permittivity and the permeability are denoted as primed or double-primed terms respectively. The "−" sign in the imaginary part is compatible with the Fourier transform definition as well as with the phasors definitions (in many physics textbooks the "+" sign is selected and opposite signs in the exponents of the Fourier transforms and phasors). The doubled-primed terms denote losses that the electromagnetic field suffers as it passes through a medium at frequencies near resonances where the $\varepsilon_r''(\omega)>0$ and $\mu_r''(\omega)>0$ terms can represent absorption. When the frequency of the electromagnetic wave is far from the resonances then $\varepsilon_r''(\omega)\simeq\mu_r''(\omega)\simeq 0$ and this frequency range is generally refer to as "transparency range."

In most realistic situations even monochromatic radiation has a spectrum differing from a delta function (even lasers exhibit broadening of their spectrum). Therefore [4, 6, 5] it is common to consider the electric and magnetic fields of the form

$$\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}_0(t)\cos(\omega_0 t + \phi_e), \tag{86}$$

$$\vec{\mathcal{H}}(t) = \vec{\mathcal{H}}_0(t)\cos(\omega_0 t + \phi_h), \tag{87}$$

where for simplicity the spatial dependence is implied. The $\vec{\mathcal{E}}_0(t)$ and $\vec{\mathcal{H}}_0(t)$ are slowly varying amplitudes (their Fourier transforms do not have high frequency terms), ω_0 is the main frequency (high frequency) of oscillation of the electromagnetic field and ϕ_e, ϕ_h are phase constants which can be space dependent. Using the approach of Jackson [4] it can be shown that

$$\left\langle \vec{\mathcal{E}}\cdot\frac{\partial\vec{\mathcal{D}}}{\partial t} + \vec{\mathcal{H}}\cdot\frac{\partial\vec{\mathcal{B}}}{\partial t}\right\rangle_{HF} = \omega_0\epsilon''(\omega_0)\langle\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}\rangle_{HF} + \omega_0\mu''(\omega_0)\langle\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}\rangle_{HF} + \frac{\partial u_{eff}}{\partial t}, \tag{88}$$

$$u_{eff} = \frac{1}{2}\frac{d}{d\omega}\Big(\omega\epsilon'(\omega)\Big)|_{\omega_0}\left\langle\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}\right\rangle_{HF} + \frac{1}{2}\frac{d}{d\omega}\Big(\omega\mu'(\omega)\Big)|_{\omega_0}\left\langle\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}\right\rangle_{HF}, \tag{89}$$

$$\left\langle\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}\right\rangle_{HF} = \frac{1}{2}\vec{\mathcal{E}}_0\cdot\vec{\mathcal{E}}_0, \tag{90}$$

$$\left\langle\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}\right\rangle_{HF} = \frac{1}{2}\vec{\mathcal{H}}_0\cdot\vec{\mathcal{H}}_0, \tag{91}$$

where $\langle * \rangle_{HF}$ denotes a high-frequency averaging (over a time period $2\pi/\omega_0$). The term u_{eff} denotes the effective electromagnetic energy density while the first two terms of the right hand side of Eq. (88) denote dissipation losses. The differential form of the Poynting's theorem can then be written as

$$-\vec{\nabla}\cdot\left(\left\langle\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right\rangle_{HF}\right) = \left\langle\vec{\mathcal{E}}\cdot\vec{\mathcal{J}}\right\rangle_{HF} + \omega_0\epsilon''(\omega_0)\left\langle\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}\right\rangle_{HF} + \omega_0\mu''(\omega_0)\left\langle\vec{\mathcal{H}}\cdot\vec{\mathcal{H}}\right\rangle_{HF} + \frac{\partial u_{eff}}{\partial t}. \tag{92}$$

At this point the time-harmonic form of the Poynting theorem will be reviewed. All the field components are represented by phasors. For example, the electric field $\vec{\mathcal{E}}$ and its phasor $\vec{E}$ are related by $\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\{\vec{E}(\vec{r},\omega)\exp(j\omega t)\}$. The time average Poynting vector is given by

$$\left\langle\vec{\mathcal{N}}\right\rangle = \frac{1}{T}\int_0^T \vec{\mathcal{N}}dt = \frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\vec{H}^*\right\} = \mathrm{Re}\{\vec{S}\}, \tag{93}$$

where $\vec{S} = (1/2)\vec{E}\times\vec{H}^*$ is the complex Poynting vector. Manipulating Maxwell's equations in a similar manner as for the derivation of Eq. (77) it can be easily shown that

$$-\vec{\nabla}\cdot\vec{S} = \frac{1}{2}\vec{E}\cdot\vec{J}^* - \frac{1}{2}j\omega\vec{E}\cdot\vec{D}^* + \frac{1}{2}j\omega\vec{H}^*\cdot\vec{B}. \tag{94}$$

Using Eqs. (24), (25), (26), (84), and (85), and $\vec{J} = \vec{J}_s + \sigma(\omega)\vec{E}$ (where $\vec{J}_s$ represents source currents) the previous equation can be written as follows (the conductivity $\sigma(\omega)$ is assumed real)

$$-\vec{\nabla}\cdot\vec{S} = \frac{1}{2}\vec{E}\cdot\vec{J}_s^* + \frac{1}{2}\sigma(\omega)|\vec{E}|^2 + \frac{1}{2}\omega\epsilon''(\omega)|\vec{E}|^2 + \frac{1}{2}\omega\mu''(\omega)|\vec{H}|^2 + \frac{j}{2}\omega[-\epsilon'(\omega)|\vec{E}|^2 + \mu'(\omega)|\vec{H}|^2]. \tag{95}$$

The above equation is the Poynting theorem in its differential form for time harmonic fields. The Poynting theorem can expressed in its integral form by integrating Eq. (95) over a closed surface $\mathcal{S}$ surrounding a volume V. The resulting equation is

$$\begin{aligned}
-\oint\!\!\oint_S \vec{S}\cdot d\vec{\mathcal{S}} \;&=\; \frac{1}{2}\iiint_V \vec{E}\cdot\vec{J}_s^* dV + \frac{1}{2}\iiint_V \sigma(\omega)|\vec{E}|^2 dV + \\
&\quad \frac{1}{2}\iiint_V \omega\Big[\epsilon''(\omega)|\vec{E}|^2 + \mu''(\omega)|\vec{H}|^2\Big] dV + \\
&\quad j\frac{\omega}{2}\iiint_V \Big[-\epsilon'(\omega)|\vec{E}|^2 + \mu'(\omega)|\vec{H}|^2\Big] dV. \qquad (96)
\end{aligned}$$

From the above equation the real and imaginary parts showing the following power equilibrium

$$P + P_\ell + P_{dis} = P_s, \qquad (97)$$

$$Q + Q_{em} = Q_s, \qquad (98)$$

$$\begin{aligned}
&P = \mathrm{Re}\Big\{\oint\!\!\oint_S \vec{S}\cdot d\vec{\mathcal{S}}\Big\},\\
&P_\ell = \frac{1}{2}\iiint_V \sigma(\omega)|\vec{E}|^2 dV,\\
&P_{dis} = \frac{1}{2}\iiint_V \omega\Big[\epsilon''(\omega)|\vec{E}|^2 + \mu''(\omega)|\vec{H}|^2\Big] dV +\\
&P_s = \frac{1}{2}\iiint_V \mathrm{Re}\Big\{-\vec{E}\cdot\vec{J}_s^*\Big\} dV,\\
&Q = \Im m\Big\{\oint\!\!\oint_S \vec{S}\cdot d\vec{\mathcal{S}}\Big\},\\
&Q_{em} = \frac{\omega}{2}\iiint_V \Big[-\epsilon'(\omega)|\vec{E}|^2 + \mu'(\omega)|\vec{H}|^2\Big] dV.\\
&Q_s = \frac{1}{2}\iiint_V \Im m\Big\{-\vec{E}\cdot\vec{J}_s^*\Big\} dV,
\end{aligned}$$

where Eq. (97) is the real power equilibrium, with P is the power exiting volume V, P_ℓ is the power consumed in ohmic losses (conduction currents) inside V, P_{dis} is the power that is dissipated into volume V due to material dielectric and magnetic losses, P_s is the power delivered by the sources in volume V. Equation (98) reveals the equilibrium of reactive power, with Q is the reactive power exiting volume V, Q_{em} is the reactive power stored in volume V (with a capacitive part due to electric field and an inductive part due to the magnetic field), and Q_s is the reactive power delivered by the sources in volume V.

Summarizing the time-averaged energy densities in the case of dispersive media can be written as

$$\langle w_e\rangle \;=\; \frac{1}{4}\frac{d}{d\omega}\Big(\omega\epsilon'(\omega)\Big)\Big\langle\vec{\mathcal{E}}_0\cdot\vec{\mathcal{E}}_0\Big\rangle = \frac{1}{4}\frac{d}{d\omega}\Big(\omega\epsilon'(\omega)\Big)\mathrm{Re}\{\vec{E}\cdot\vec{E}^*\}, \qquad (99)$$

$$\langle w_m\rangle \;=\; \frac{1}{4}\frac{d}{d\omega}\Big(\omega\mu'(\omega)\Big)\Big\langle\vec{\mathcal{H}}_0\cdot\vec{\mathcal{H}}_0\Big\rangle = \frac{1}{4}\frac{d}{d\omega}\Big(\omega\mu'(\omega)\Big)\mathrm{Re}\{\vec{H}\cdot\vec{H}^*\}, \qquad (100)$$

while in the time harmonic case the corresponding equations become

$$\langle w_e \rangle = \frac{1}{4}\mathrm{Re}\{\vec{E}\cdot\vec{D}^*\}, \tag{101}$$

$$\langle w_m \rangle = \frac{1}{4}\mathrm{Re}\{\vec{H}\cdot\vec{B}^*\}. \tag{102}$$

In the case of a plane electromagnetic wave the time-averaged electric energy density is equal to the time-averaged magnetic energy density, and total electromagnetic energy density is given by

$$\langle w_{em} \rangle = 2\langle w_e \rangle = 2\langle w_m \rangle = \frac{1}{2}\mathrm{Re}\{\vec{E}\cdot\vec{D}^*\} = \frac{1}{2}\mathrm{Re}\{\vec{H}\cdot\vec{B}^*\}. \tag{103}$$

When a plane wave is incident at a planar boundary between two linear, isotropic, lossless, and non-magnetic regions application of Poynting theorem gives the following power conservation equation for the direction normal to the boundary

$$\frac{P_r}{P_i} + \frac{P_t}{P_i} = |r_{TE}|^2 + |t_{TE}|^2 \frac{\mathrm{Re}\{n_2 \cos\theta_2\}}{n_1 \cos\theta_1} = 1, \qquad \text{TE Polarization} \tag{104}$$

$$\frac{P_r}{P_i} + \frac{P_t}{P_i} = |r_{TM}|^2 + |t_{TM}|^2 \frac{\mathrm{Re}\{n_2^* \cos\theta_2\}}{n_1 \cos\theta_1} = 1, \qquad \text{TM Polarization,} \tag{105}$$

where P_i, P_r, P_t are the indident, reflected, and transmitted powers. In the case of lossless medium in region 2 the above equations are the same. The term n_2^* denote the complex conjugate of the refractive index and of the complex cosine. These are in general complex in the case that region 2 consists of a lossy material such as a metal. The last equations simply state that the percentage of the reflected power plus the percentage of the transmitted power equals unity for a lossless case of reflection/refraction at a planar boundary.

5. Generalized Reflection and Transmission at a Planar Boundary

The geometry of the planar interface between to isotropic media is again depicted in Fig. 8. In this case the polarization of the incident wave has both TE (perpendicular) and TM (parallel) components. Therefore, in this formulation any elliptical in general polarization of the incident wave can be treated. The corresponding incident wave can be described by its electric field phasor

$$\vec{E}_i = E_{TE}\hat{u}_{TE}\exp(-j\vec{k}_i\cdot\vec{r}) + E_{TM}\hat{u}_{TM}\exp(-j\vec{k}_i\cdot\vec{r}), \quad \text{where} \tag{106}$$

$$\hat{u}_{TE} = \hat{y},$$

$$\hat{u}_{TM} = \cos\theta_1\ \hat{x} - \sin\theta_1\ \hat{z},$$

$$\vec{k}_i = k_0\sqrt{\epsilon_{r1}\mu_{r1}}\left(\sin\theta_1\ \hat{x} + \cos\theta_1\ \hat{z}\right),$$

and ϵ_{r1}, μ_{r1} are the relative permittivity and relative permeability of region 1 (in case of non-magnetic medium $\mu_{r1} = 1$). The E_{TE} and E_{TM} complex amplitude terms can represent any, elliptical in general, polarization. The reflected electric field phasor is given by

$$\begin{aligned} \vec{E}_r &= E_{TE} r_{TE} \hat{u}^r_{TE} \exp(-j\vec{k}_r \cdot \vec{r}) + E_{TM} r_{TM} \hat{u}^r_{TM} \exp(-j\vec{k}_r \cdot \vec{r}), \quad \text{where} \\ &\hat{u}^r_{TE} = \hat{u}_{TE} = \hat{y}, \\ &\hat{u}^r_{TM} = -\cos\theta_1 \ \hat{x} - \sin\theta_1 \ \hat{z}, \\ &\vec{k}_r = k_0 \sqrt{\epsilon_{r1}\mu_{r1}} \left(\sin\theta_1 \ \hat{x} - \cos\theta_1 \ \hat{z} \right), \end{aligned} \tag{107}$$

where r_{TE} and r_{TM} are the amplitude reflection coefficients (complex in general). Therefore, the total electric field phasor of region 1 is given by

$$\begin{aligned} \vec{E}_1 &= E_{TE} \hat{u}_{TE} \exp(-j\vec{k}_i \cdot \vec{r}) + E_{TM} \hat{u}_{TM} \exp(-j\vec{k}_i \cdot \vec{r}) + \\ &\quad E_{TE} r_{TE} \hat{u}_{TE} \exp(-j\vec{k}_r \cdot \vec{r}) + E_{TM} r_{TM} \hat{u}^r_{TM} \exp(-j\vec{k}_r \cdot \vec{r}). \end{aligned} \tag{108}$$

The corresponding magnetic field phasor in region 1 can straightforwardly be determined and is given by

$$\begin{aligned} \vec{H}_1 &= -\frac{E_{TE}}{Z_1} \hat{u}_{TM} \exp(-j\vec{k}_i \cdot \vec{r}) + \frac{E_{TM}}{Z_1} \hat{u}_{TE} \exp(-j\vec{k}_i \cdot \vec{r}) - \\ &\quad \frac{E_{TE} r_{TE}}{Z_1} \hat{u}^r_{TM} \exp(-j\vec{k}_r \cdot \vec{r}) + \frac{E_{TM} r_{TM}}{Z_1} \hat{u}_{TE} \exp(-j\vec{k}_r \cdot \vec{r}), \end{aligned} \tag{109}$$

where $Z_1 = (\mu_1/\epsilon_1)^{1/2}$ is the wave impedance of region 1. The electric and magnetic field phasors in region 2 are similarly given by

$$\vec{E}_2 = t_{TE} E_{TE} \hat{u}_{TE} \exp(-j\vec{k}_t \cdot \vec{r}) + t_{TM} E_{TM} \hat{u}^t_{TM} \exp(-j\vec{k}_t \cdot \vec{r}), \tag{110}$$

$$\begin{aligned} \vec{H}_2 &= -\frac{t_{TE} E_{TE}}{Z_2} \hat{u}^t_{TM} \exp(-j\vec{k}_t \cdot \vec{r}) + \frac{t_{TM} E_{TM}}{Z_2} \hat{u}_{TE} \exp(-j\vec{k}_t \cdot \vec{r}), \\ &\vec{k}_t = k_0 \sqrt{\epsilon_{r2}\mu_{r2}} \left(\sin\theta_2 \ \hat{x} + \cos\theta_2 \ \hat{z} \right), \\ &\hat{u}^t_{TM} = \cos\theta_2 \ \hat{x} - \sin\theta_2 \ \hat{z}, \end{aligned} \tag{111}$$

where ϵ_{r2}, μ_{r2} are the relative permittivity and relative permeability of region 2 (in case of nonmagnetic medium $\mu_{r2} = 1$), and t_{TE} and t_{TM} are the amplitude transmission coefficients (complex in general).

Applying the boundary conditions at the $z = 0$ planar interface for the tangential electric and magnetic field components it is straightforward to obtain the following equations:

$$\begin{aligned}
\sqrt{\epsilon_{r1}\mu_{r1}}\sin\theta_1 &= \sqrt{\epsilon_{r2}\mu_{r2}}\sin\theta_2, && (112)\\
1 + r_{_{TE}} &= t_{_{TE}}, && (113)\\
\cos\theta_1 - r_{_{TM}}\cos\theta_1 &= t_{_{TM}}\cos\theta_2, && (114)\\
-\frac{\cos\theta_1}{Z_1} + \frac{\cos\theta_1 r_{_{TE}}}{Z_1} &= -\frac{\cos\theta_2 t_{_{TE}}}{Z_2}, && (115)\\
\frac{1}{Z_1} + \frac{r_{_{TM}}}{Z_1} &= \frac{t_{_{TM}}}{Z_2}. && (116)
\end{aligned}$$

Solving the previous equations the amplitude reflections and transmitted coefficients can be easily calculated in this general formulation and are given by

$$r_{_{TE}} = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}, \quad (117)$$

$$t_{_{TE}} = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}, \quad (118)$$

$$r_{_{TM}} = \frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}, \quad (119)$$

$$t_{_{TM}} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2}, \quad (120)$$

where the last equations are generalized equivalent to Eqs. (65), (66), (67), and (68) which hold for nonmagnetic media. It is mentioned that region 2 can be lossy and in the latter case ϵ_2 and μ_2 can become complex.

In order to evaluate the percentage of the power reflected and the percentage of the power transmitted the z-components of the complex Poynting vectors must be evaluated. Using the phasor fields defined previously, after some manipulations, the following z-components of the complex Poynting vectors for regions 1 and 2 can be determined:

$$\begin{aligned}
S_{1z} = \frac{\cos\theta_1}{Z_1}\Big\{ & |E_{_{TE}}|^2 + |E_{_{TM}}|^2 - |E_{_{TE}}|^2|r_{_{TE}}|^2 - |E_{_{TM}}|^2|r_{_{TM}}|^2 - \\
& |E_{_{TE}}|^2 r^*_{_{TE}}\exp\left(+j(\vec{k}_r - \vec{k}_i)\cdot\vec{r}\right) + |E_{_{TE}}|^2 r_{_{TE}}\exp\left(-j(\vec{k}_r - \vec{k}_i)\cdot\vec{r}\right) + \\
& |E_{_{TM}}|^2 r^*_{_{TM}}\exp\left(+j(\vec{k}_r - \vec{k}_i)\cdot\vec{r}\right) - |E_{_{TM}}|^2 r_{_{TM}}\exp\left(-j(\vec{k}_r - \vec{k}_i)\cdot\vec{r}\right)\Big\} \quad (121)\\
S_{2z} = \frac{1}{Z_2^*}\Big\{ & |t_{_{TE}}|^2|E_{_{TE}}|^2(\cos\theta_2)^* + |t_{_{TM}}|^2|E_{_{TM}}|^2\cos\theta_2\Big\}. \quad (122)
\end{aligned}$$

Equating the real parts of the z-components of the Poynting vectors the normalized reflected and transmitted powers are given by

$$\frac{P_r}{P_i} = \frac{|r_{TE}|^2|E_{TE}|^2 + |r_{TM}|^2|E_{TM}|^2}{|E_{TE}|^2 + |E_{TM}|^2}, \tag{123}$$

$$\frac{P_t}{P_i} = \frac{Z_1}{\cos\theta_1} \frac{|t_{TE}|^2|E_{TE}|^2\mathrm{Re}\{(\cos\theta_2)^*/Z_2^*)\} + |t_{TM}|^2|E_{TM}|^2\mathrm{Re}\{\cos\theta_2/Z_2^*)\}}{|E_{TE}|^2 + |E_{TM}|^2}. \tag{124}$$

The last two equations take the simple form of Eqs. (104) and (105) in the case of the TE ($E_{TM} = 0$) or TM ($E_{TE} = 0$) polarization respectively.

6. Jones Calculus

A simple and systematic approach for analyzing the effects of light passing through a system of anisotropic plates and polarizers is "Jones calculus" that was invented by R. Clark Jones in 1941 [7]. In Jones' approach the polarization state is represented by a two-component vector and each optical element is represented by a 2×2 matrix. The overall system response is obtained by multiplication of all the individual 2×2 matrices and the output polarization state is determined by multiplying the input polarization state-vector with the overall optical system 2×2 matrix.

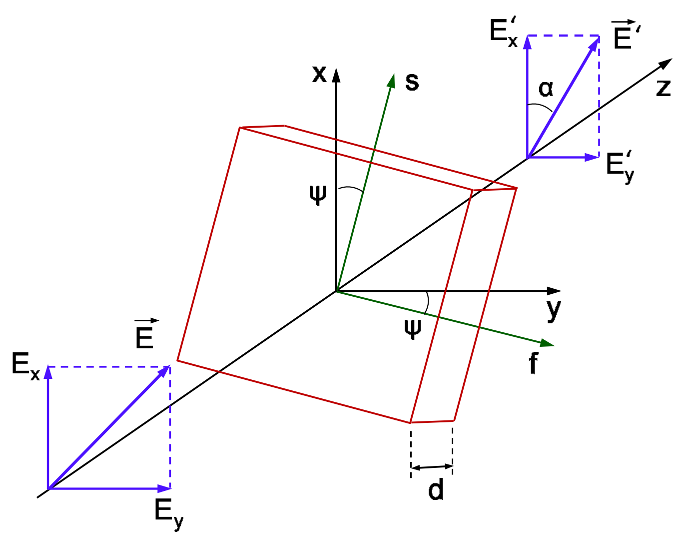


Figure 15: A retardation plate with principal axes s (slow), f (fast), and z, is shown in the laboratory coordinate system xyz. The slow and fast axes are at an angle ψ with respect to x and y axes respectively. The thickness of the plate is d along the z direction. It is assumed that the angle psi is measured positive along the right-handed direction.

When light propagates in anisotropic media it consists of a linear superposition of two orthogonally polarized waves, the eigen-waves, with their corresponding eigen-polarizations and refractive indices. These waves are well defined for each direction of propagation (remember the index ellipsoid approach). For the most common retardation plates one of the principal axis is along the z direction (as shown in Fig. 15). In this case the other two principal axes (named as slow axis and fast axis) lie in the plane of the plate (xy plane in Fig. 15). In Jones calculus approach all the reflections as well as all multiple interference effects are neglected. This is usually a reasonable approximation since the main surfaces of these plates are commonly coated with antireflective layers. Furthermore, the plate main surfaces are not usually optically parallel which justifies the neglect of the multiple interference effects.

The polarization state of the incident electromagnetic wave can be represented as follows

$$\vec{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix}, \tag{125}$$

which is compatible with the decomposition of the incident field shown in Fig. 15. This incident wave should be decomposed into the eigen-polarizations (along f and s axes). This can be accomplished as follows

$$\begin{bmatrix} E_s \\ E_f \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \mathcal{R}(\psi) \begin{bmatrix} E_x \\ E_y \end{bmatrix}. \tag{126}$$

The eigen-polarizations E_s and E_f propagate into the plate of thickness d independent of each other according to the equation

$$\begin{bmatrix} E_s' \\ E_f' \end{bmatrix} = \begin{bmatrix} \exp(-jk_0 n_s d) & 0 \\ 0 & \exp(-jk_0 n_f d) \end{bmatrix} \begin{bmatrix} E_s \\ E_f \end{bmatrix}, \tag{127}$$

where the primed values E_s' and E_f' denotes the eigen-polarizations at the end of the propagation distance d, n_s and n_f are the slow and fast axis refractive indices, and k_0 the freespace wavenumber. Now if the retardation $\Gamma = k_0(n_s - n_f)d$ and the phase angle $\Phi = (1/2)k_0(n_s + n_f)d$ are used the previous equation can be written as

$$\begin{bmatrix} E_s' \\ E_f' \end{bmatrix} = \exp(-j\Phi) \begin{bmatrix} \exp(-j\Gamma/2) & 0 \\ 0 & \exp(+j\Gamma/2) \end{bmatrix} \begin{bmatrix} E_s \\ E_f \end{bmatrix}, \tag{128}$$

and converting the last equation into the xy coordinate system it results in

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_s' \\ E_f' \end{bmatrix} = \mathcal{R}(-\psi) \begin{bmatrix} E_s' \\ E_f' \end{bmatrix}, \tag{129}$$

where the primed fields are the electric field components at the exit from the plate expressed in the xy coordinate system. Combining all the steps together it is possible to relate the output electric field components E_x' and E_y' with the input ones E_x and E_y.

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} = \mathcal{R}(-\psi) \exp(-j\Phi) \begin{bmatrix} \exp(-j\Gamma/2) & 0 \\ 0 & \exp(+j\Gamma/2) \end{bmatrix} \mathcal{R}(\psi) \begin{bmatrix} E_x \\ E_y \end{bmatrix} = W(\psi, \Gamma) \begin{bmatrix} E_x \\ E_y \end{bmatrix}, \tag{130}$$

where the overall 2×2 matrix $W(\psi, \Gamma)$ is given by

$$W(\psi,\Gamma) = e^{-j\Phi} \begin{bmatrix} e^{-j\frac{\Gamma}{2}}\cos^2\psi + e^{+j\frac{\Gamma}{2}}\sin^2\psi & -j\sin\frac{\Gamma}{2}\sin 2\psi \\ -j\sin\frac{\Gamma}{2}\sin 2\psi & e^{-j\frac{\Gamma}{2}}\sin^2\psi + e^{+j\frac{\Gamma}{2}}\cos^2\psi \end{bmatrix}. \quad (131)$$

It is important to note that since the plate is lossless (and no reflections are taken into account) the electromagnetic power density (Poynting vector) will remain constant after passing through the plate. I.e. $|E'_x|^2 + |E'_y|^2 = |E_x|^2 + |E_y|^2$, which is something that can be easily verified from Eq. (130).

An ideal polarizer with its transmission axis parallel to the x axis can be represented by the following Jones matrix

$$P_x = e^{-j\Theta} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (132)$$

while an ideal polarizer along the y axis is represented by the Jones matrix

$$P_y = e^{-j\Theta} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (133)$$

where Θ represents a phase shift accumulated due to the finite size of the plate along the z axis. Finally, an ideal polarizer with its transmission axis oriented at an angle Ψ with respect to the x axis can be described by the following Jones matrix

$$P(\Psi) = e^{-j\Theta} \begin{bmatrix} \cos^2\Psi & -\sin\Psi\cos\Psi \\ -\sin\Psi\cos\Psi & \sin^2\Psi \end{bmatrix}, \quad (134)$$

where $P(\Psi = 0) = P_x$ and $P(\Psi = \pi/2) = P_y$ correspond to special cases of polarizers oriented along the x and the y axis, respectively.

Example 1: A half-wave plate is considered. By definition the half-wave plate has a retardation $\Gamma = \pi = k_0(n_s - n_f)d$. It is also assumed that $\psi = \pi/4$ (i.e. the slow axis is at 45 degrees with respect to the x axis). The incident plane wave is polarized along the y axis, i.e. $E_x = 0$ and $E_y = E_0 \neq 0$. The corresponding Jones matrix is given by Eq. (131) for $\Gamma = \pi$ and $\psi = \pi/4$ and is equal to

$$W(\Gamma = \pi, \psi = \frac{\pi}{4}) = e^{-j\Phi} \begin{bmatrix} 0 & -j \\ -j & 0 \end{bmatrix}. \quad (135)$$

The resulting field at the output of the half-wave plate is

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = e^{-j\Phi} \begin{bmatrix} 0 & -j \\ -j & 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_0 \end{bmatrix} = e^{-j\Phi} \begin{bmatrix} -jE_0 \\ 0 \end{bmatrix} = -je^{-j\Phi} \begin{bmatrix} E_0 \\ 0 \end{bmatrix}. \quad (136)$$

The last equation reveals that the y-polarized plane wave was transformed into an x-polarized plane wave. If the incident polarization is right-handed circularly polarized then $[E_x, E_y]^T = E_0[1, -j]^T$ and applying again Eq. (131) it is straightforward to show that $[E'_x, E'_y]^T = E_0[1, +j]^T$ which corresponds to a left-handed circularly polarized plane wave.

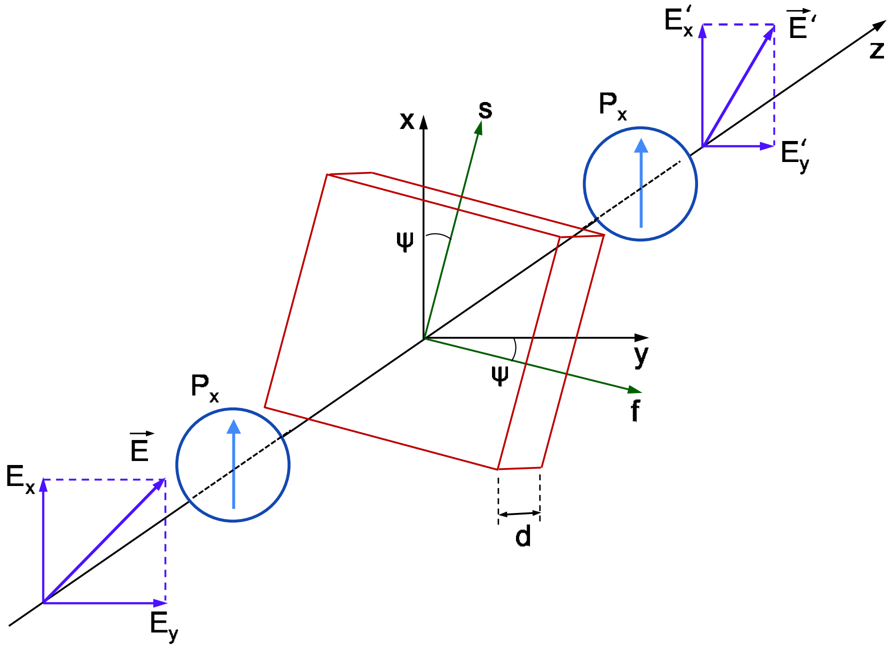


Figure 16: A retardation plate with principal axes s (slow), f (fast), and z, is shown in the laboratory coordinate system xyz. The slow and fast axes are at an angle ψ with respect to x and y axes respectively. The thickness of the plate is d along the z direction. Two parallel x-polarizers are inserted in front of the plate and after the plate. This system can cause intensity modulation of the incident plane wave.

Example 2: In this example a retardation plate is sandwiched between two x polarizers as shown in Fig. 16. The angle $\psi = \pi/4$. Assume that the incident wave is polarized along the x axis (parallel to the polarizer).

Then the resulting polarization vector after the output polarizer is given by

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = e^{-j(\Phi+\Theta)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2} \\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-j(\Phi+\Theta)} \begin{bmatrix} \cos\frac{\Gamma}{2} \\ 0 \end{bmatrix}. \tag{137}$$

From the last equation the output intensity can be related to the input intensity as follows

$$\frac{I_{out}}{I_{in}} = \frac{|E'_x|^2 + |E'_y|^2}{|E_x|^2 + |E_y|^2} = \cos^2\frac{\Gamma}{2}. \tag{138}$$

The last equation represents modulation of the output intensity according to the retardation angle Γ. This can be accomplished by varying Γ through the electro-optic effect.

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