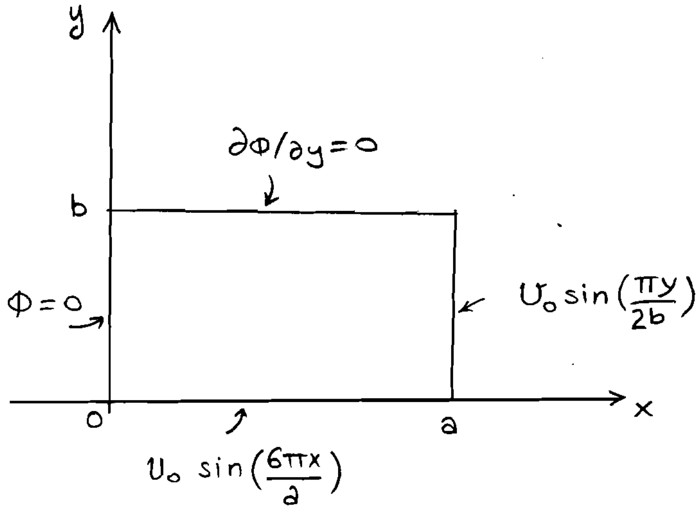
ΣΕΙΡΑ ΑΣΚΗΣΕΩΝ Νο. 4

ΑΣΚΗΣΗ 1:

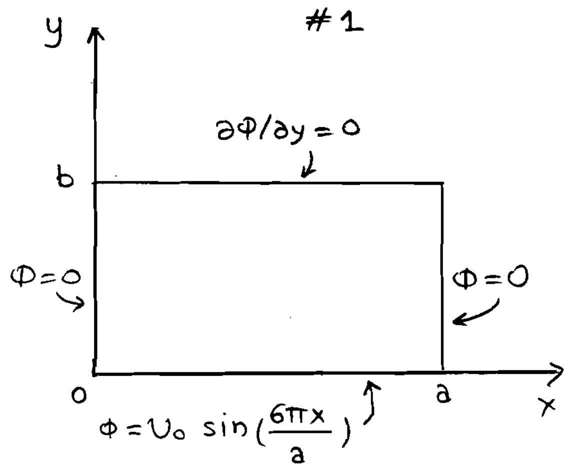


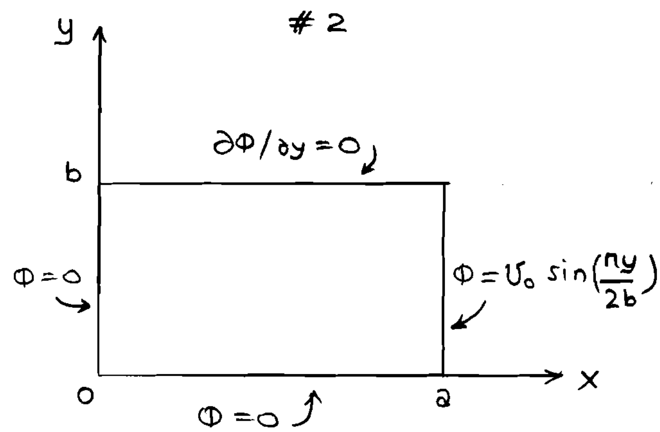
Ας εξετάσουμε πρώτα την συνέχεια των οριακών συνθηκών:

$$\Phi(x \to 0, 0) = 0 = \Phi(0, y \to 0)$$

$$\Phi(x \to a, 0) = 0 = \Phi(a, y \to 0) = 0$$

Το πρόβλημα μπορεί να αναλυθεί στα εξής επιμέρους προβλήματα.





Πρόβλημα #1:

Εφόσον για $0<x<a$ εμφανίζεται περιοδική λύση στο $y=0$ διαλέγουμε την λύση:

$$\Phi_1(x,y) = \left[A_1 \sin(kx) + A_2 \cos(kx)\right]\left[B_1 e^{ky} + B_2 e^{-ky}\right]$$

$$\Phi_1(x=0, y) = 0 \Rightarrow \quad A_2 = 0$$

$$\Phi_1(x=a, y) = 0 \Rightarrow \quad A_1' \sin(ka) = 0 \to ka = n\pi \Rightarrow k = \frac{n\pi}{a}$$

$$\Phi_1(x, y=0) = U_0 \sin\left(\frac{6\pi x}{a}\right) = A_1 \sin\left(\frac{n\pi}{a}x\right)[B_1 + B_2]$$

Ο συντελεστής A_1 μπορεί να ενσωματωθεί με τα B_1, B_2.

Επομένως $n = 6$ και $B_1 + B_2 = U_0$ και $k = \frac{6\pi}{a}$

$$\left.\frac{\partial \Phi_1}{\partial y}\right|_{y=b} = 0 \Rightarrow \sin\left(\frac{6\pi x}{a}\right)\left[\frac{6\pi}{a} B_1 e^{ky} - \frac{6\pi}{a} B_2 e^{-ky}\right]\Bigg|_{y=b} = 0$$

$$B_1 e^{kb} - B_2 e^{-kb} = 0 \Rightarrow \left.\begin{array}{l} B_1 e^{2kb} = B_2 \\ B_1 + B_2 = U_0 \end{array}\right\} \rightarrow \begin{array}{l} B_1 = \dfrac{U_0}{1+e^{2kb}} \\ B_2 = \dfrac{U_0 e^{2kb}}{1+e^{2kb}} \end{array}$$

$$k = 6\pi/a$$

Επομένως η λύση του προβλήματος 1 είναι:

$$\Phi_1(x,y) = \sin\left(\frac{6\pi x}{a}\right)\left[\frac{U_0}{1+e^{2kb}} e^{ky} + \frac{U_0 e^{2kb}}{1+e^{2kb}} e^{-ky}\right]$$

$$= U_0 \sin\left(\frac{6\pi x}{a}\right) \frac{e^{k(y-b)} + e^{-k(y-b)}}{e^{kb} + e^{-kb}} =$$

$$= U_0 \sin\left(\frac{6\pi x}{a}\right) \frac{\cosh\left[\frac{6\pi}{a}(y-b)\right]}{\cosh\left(\frac{6\pi}{a} b\right)}$$

Πρόβλημα #2:

Τώρα έχουμε περιοδική λύση για y στο $x=a$. Έτσι επιλέγουμε την λύση

$$\Phi_2(x,y) = [A_1 e^{kx} + A_2 e^{-kx}][B_1 \sin(ky) + B_2 \cos(ky)]$$

Οριακές Συνθήκες:

$$\Phi_2(x, y=0) = 0 \Rightarrow (A_1 e^{kx} + A_2 e^{-kx}) B_2 \cos(ky) = 0 \sim B_2 = 0$$

Επομένως $\Phi_2(x,y) = \sin(ky)[A_1 e^{kx} + A_2 e^{-kx}]$

$$\left.\frac{\partial \Phi}{\partial y}\right|_{y=b} = 0 \Rightarrow k\cos(ky)\Big|_{y=b}[A_1 e^{kx} + A_2 e^{-kx}] = 0 \Rightarrow$$

$$kb = (2m+1)\frac{\pi}{2} \Rightarrow k = (2m+1)\frac{\pi}{2b} \quad m = 0, 1, 2, \ldots$$

$$\Phi(x=0, y) = 0 \Rightarrow A_1 + A_2 = 0$$

$$\Phi(x=a, y) = U_0 \sin\left(\frac{\pi y}{2b}\right) = \sin(ky)[A_1 e^{ka} + A_2 e^{-ka}] \Rightarrow$$

$k = \frac{\pi}{2b}$ ($m=0$ από την γενική λύση).

και $U_0 = A_1 e^{ka} + A_2 e^{-ka}$

Επομένως $A_1 = \frac{U_0}{e^{ka} - e^{-ka}} = \frac{U_0}{2\sinh(ka)}$ και $A_2 = -\frac{U_0}{2\sinh(ka)}$

Άρα

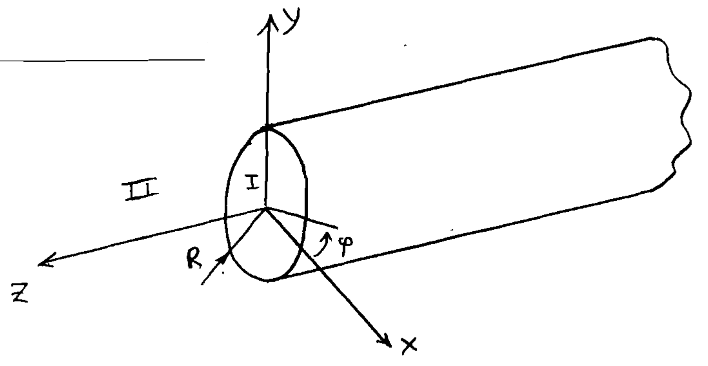
$$\Phi_2(x,y) = U_0 \sin\left(\frac{\pi}{2b} y\right) \frac{\sinh\left(\frac{\pi}{2b} x\right)}{\sinh\left(\frac{\pi}{2b} a\right)}$$

Συνοψίζοντας η λύση του αρχικού προβλήματος είναι:

$$\Phi(x,y) = \Phi_1(x,y) + \Phi_2(x,y) =$$

$$= U_0 \sin\left(\frac{6\pi x}{a}\right) \frac{\cosh\left(\frac{6\pi}{a}(y-b)\right)}{\cosh\left(\frac{6\pi}{a} b\right)} + U_0 \frac{\sinh\left(\frac{\pi}{2b} x\right)}{\sinh\left(\frac{\pi}{2b} a\right)} \sin\left(\frac{\pi}{2b} y\right)$$

ΑΣΚΗΣΗ 2:



(α) Εφόσον υπάρχει περιοδικότητα ως πρός φ $K_\varphi = -m^2$ και $K_{r_T} = +m^2$ οπότε οι λύσεις είναι της μορφής:

$$\Phi(r_T,\varphi) = \sum_m (A_m r_T^m + B_m r_T^{-m})(\Gamma_m \sin(m\varphi) + \Delta_m \cos(m\varphi))$$

ή

$$(\Gamma_m' e^{jm\varphi} + \Delta_m' e^{-jm\varphi})$$

Αν $m=0$ η λύση της μορφής $(A_0 \ln(r_T) + B_0)(\Gamma_0 \varphi + \Delta_0)$ καταλήγει σε σταθερά διότι $\Gamma_0 = A_0 = 0$ λόγω του ότι Φ πρέπει να είναι πεπερασμένο για $r_T = 0$, $r_T = \infty$. Οπότε έχουμε:

$$\Phi_I(r_T,\varphi) = \alpha_0 + \sum_{m=1}^{\infty} (a_m r_T^m + b_m r_T^{-m})(\gamma_m \sin(m\varphi) + \delta_m \cos(m\varphi))$$

αφού $r_T \to 0$ Φ_I πεπερασμένο $b_m = 0\ \forall m$ και επομένως:

$$\Phi_I(r_T,\varphi) = \alpha_0 + \sum_{m=1}^{\infty} r_T^m (\gamma_m \sin(m\varphi) + \delta_m \cos(m\varphi))$$

Παρομοίως,

$$\Phi_{II}(r_T,\varphi) = A_0 + \sum_{m=1}^{\infty} (A_m r_T^m + B_m r_T^{-m})(\Gamma_m \sin(m\varphi) + \Delta_m \cos(m\varphi))$$

και Φ_{II} πεπερασμένο $(\to 0)$ για $r_T \to \infty \Rightarrow A_m = 0\ \forall m$, οπότε

$$\Phi_{II}(r_T,\varphi) = A_0 + \sum_{m=1}^{\infty} r_T^{-m} (\Gamma_m \sin(m\varphi) + \Delta_m \cos(m\varphi))$$

(β) $\sigma = \sigma_0 \sin(5\varphi)$

$$\sigma = \hat{i}_n \cdot (\vec{D}_+ - \vec{D}_-) = \hat{i}_n \cdot (\vec{D}_{II} - \vec{D}_I) = \hat{r}_T \cdot \left[-\epsilon_0 \left(\frac{\partial \Phi_{II}}{\partial r_T} \hat{i}_{r_T} \right) + \epsilon_0 \frac{\partial \Phi_I}{\partial r_T} \hat{i}_{r_T} \right]$$

$$= -\epsilon_0 \left. \frac{\partial \Phi_{II}}{\partial r_T} \right|_R + \epsilon_0 \left. \frac{\partial \Phi_I}{\partial r_T} \right|_R$$

$$\left.\frac{\partial \Phi_I}{\partial r_T}\right|_R = \sum_m m\, r_T^{m-1} \left(\gamma_m \sin(m\varphi) + \delta_m \cos(m\varphi)\right)\Big|_{r_T=R}$$

$$\left.\frac{\partial \varphi_{II}}{\partial r_T}\right|_R = \sum_m (-m) r^{-m-1} \left(\Gamma_m \sin(m\varphi) + \Delta m \cos(m\varphi)\right)\Big|_{r_T=R}$$

Επομένως,

$$\sigma_0 \sin 5\varphi = +\epsilon_0 \Big[\sum_m \left(m R^{-m-1} \Gamma_m + m R^{m-1} \gamma_m\right) \sin(m\varphi) + \sum_m \left(m R^{-m-1} \Delta_m + m R^{m-1} \delta_m\right) \cos(m\varphi) \Big]$$

$$\frac{\sigma_0}{\epsilon_0} = \left(5 R^{-6} \Gamma_5 + 5 R^4 \gamma_5\right) \qquad m=5$$

$$0 = m R^{-m-1} \Gamma_m + m R^{m-1} \gamma_m \qquad \forall m \quad m \neq 5$$

$$0 = m R^{-m-1} \Delta_m + m R^{m-1} \delta_m \qquad \forall m$$

Επιπλέον $\Phi_I (r_T=R, \varphi) = \Phi_{II} (r_T=R, \varphi) \rightsquigarrow$

$$\alpha_0 + \sum_m R^m \left(\gamma_m \sin(m\varphi) + \delta_m \cos(m\varphi)\right) =$$

$$A_0 + \sum_m R^{-m} \left(\Gamma_m \sin(m\varphi) + \Delta m \cos(m\varphi)\right)$$

Επομένως $\alpha_0 = A_0 = 0$ χωρίς πρόβλημα μια και το δυναμικό ορίζεται ως προς μια σταθερά.

$$R^m \gamma_m = R^{-m} \Gamma_m$$

$$R^m \delta_m = R^{-m} \Delta_m$$

Επομένως $m R^{-1} R^m \delta_m + m R^{m-1} \delta_m = 0 \rightarrow \delta_m = 0 \rightsquigarrow \Delta_m = 0$

Επίσης $m R^{-1} R^m \gamma_m + m R^{m-1} \gamma_m = 0 \quad \forall m \quad m \neq 5 \rightsquigarrow$

$\gamma_m = \Gamma_m = 0 \quad \forall m$ με $m \neq 5$.

Άρα μόνο $\gamma_5, \Gamma_5 \neq 0$.

$$\frac{\sigma_0}{\epsilon_0} = 5\left(R^{-1}R^5\gamma_s + R^4\gamma_s\right) \rightsquigarrow \quad \gamma_s = \frac{\sigma_0}{10\epsilon_0}\frac{1}{R^4}$$

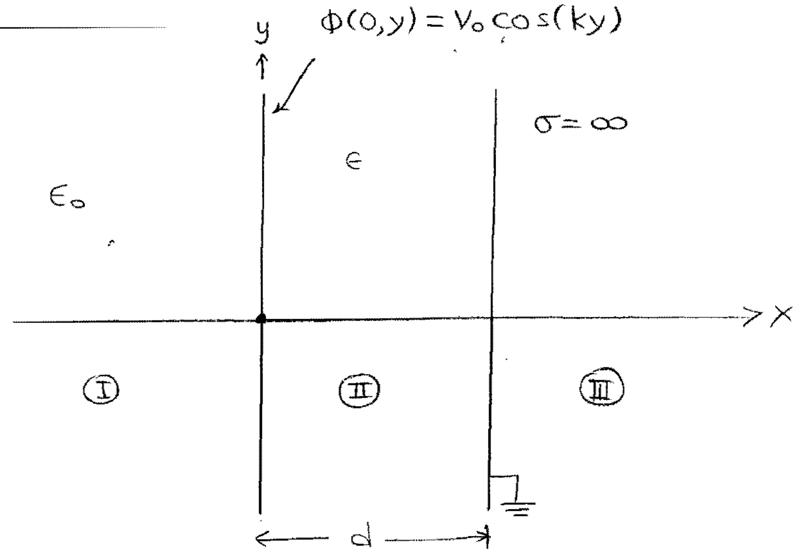
και $$\Gamma_s = R^{10}\frac{\sigma_0}{10\epsilon_0}\frac{1}{R^4} = R^6\frac{\sigma_0}{10\epsilon_0}$$

Επομένως

$$\Phi_I(r_T,\varphi) = r_T^5\,\frac{\sigma_0}{10\epsilon_0}\frac{1}{R^4}\sin(5\varphi) \qquad 0<r_T\leq R$$

$$\Phi_{II}(r_T,\varphi) = r_T^{-5}\,\frac{\sigma_0}{10\epsilon_0}R^6\sin(5\varphi) \qquad r_T\geq R$$

ΑΣΚΗΣΗ 3:



(α) Μέθοδος χωρισμού μεταβλητών:

$$\Phi(x,y) = X(x)\,Y(y)$$

Εφόσον $\Phi(0,y) = V_0 \cos ky$ είναι λογικό να επιλέγει η λύση:

$$\Phi_I(x,y) = \left[A_1 \cos(ky) + B_1 \sin(ky) \right] \left[\Gamma_1 e^{-kx} + \Delta_1 e^{+kx} \right]$$

$$\Phi_{II}(x,y) = \left[A_2 \cos(ky) + B_2 \sin(ky) \right] \left[\Gamma_2 e^{-kx} + \Delta_2 e^{+kx} \right]$$

$$\Phi_{III}(x,y) = 0$$

Αφού $\Phi_I(x=0,y) = \Phi_{II}(x=0,y) = V_0 \cos(ky)$ έχουμε:

$$\left(A_2 \cos(ky) + B_2 \sin(ky)\right)\left(\Gamma_2 + \Delta_2\right) = V_0 \cos(ky) \Rightarrow$$

$B_2 = 0$ και άρα

$$\Phi_{II}(x,y) = \cos(ky)\left[\Gamma_2 e^{kx} + \Delta_2 e^{-kx} \right] \quad \text{και} \quad \Gamma_2 + \Delta_2 = V_0$$

Όμως $\Phi_{II}(x=d, y) = 0 \Rightarrow$

$$\cos(ky)\left[\Gamma_2 e^{kd} + \Delta_2 e^{-kd} \right] = 0 \Rightarrow$$

$$\left.\begin{aligned} \Gamma_2 e^{kd} + \Delta_2 e^{-kd} &= 0 \\ \Gamma_2 + \Delta_2 &= V_0 \end{aligned}\right\} \Rightarrow \quad \Gamma_2 = -\frac{V_0 e^{-kd}}{e^{kd} - e^{-kd}}, \quad \Delta_2 = \frac{V_0 e^{kd}}{e^{kd} - e^{-kd}}$$

Επομένως,

$$\Phi_{II}(x,y) = -V_o \cos ky \, \frac{\sinh(k(x-d))}{\sinh(kd)}$$

Επιπλέον,

$$\Phi_I(0,y) = \Phi_{II}(0,y) = V_o \cos(ky)$$

Όμως $\Phi_I(x\to -\infty, y) \to 0 \rightsquigarrow \Gamma_1 \equiv 0 \rightsquigarrow$

$\cos(ky)\,\Delta_1 = V_o \cos(ky) \rightsquigarrow \Delta_1 = V_o$ και επομένως,

$$\Phi_I(x,y) = V_o \cos(ky)\, e^{kx}$$

Άρα

$$\Phi(x,y) = \begin{cases} V_o \cos(ky) e^{kx} & -\infty < x \leq 0 \quad \forall y \\ -V_o \cos(ky) \dfrac{\sinh(k(x-d))}{\sinh(kd)} & 0 \leq x \leq d \quad \forall y \\ 0 \qquad x \geq d & \forall y \end{cases}$$

(β) $\vec{E} = -\vec{\nabla}\Phi = -\dfrac{\partial\Phi}{\partial x}\hat{i}_x - \dfrac{\partial\Phi}{\partial y}\hat{i}_y$

Άρα
$$\vec{E} = \begin{cases} -kV_o \cos(ky) e^{+kx}\,\hat{i}_x + kV_o \sin(ky) e^{kx}\,\hat{i}_y & -\infty<x<0 \\ V_o \cos(ky) \dfrac{1}{\sinh(kd)}\, k\cosh(k(x-d))\,\hat{i}_x \\ -V_o k \sin(ky) \dfrac{\sinh(k(x-d))}{\sinh(kd)}\,\hat{i}_y & 0<x<d \\ 0 \qquad x>d \end{cases}$$

(γ) $\sigma(x=0,y) = \hat{i}_n \circ (\vec{D}(x=0+) - \vec{D}(x=0-)) =$

$= \hat{i}_x \circ [\hat{i}_x \epsilon E_x(0+) - \hat{i}_x \epsilon_0 E_x(0-)] =$

$= \epsilon E_x(0+) - \epsilon_0 E_x(0-) =$

$= \epsilon k V_o \cos(ky) \dfrac{\cosh(kd)}{\sinh(kd)} + \epsilon_0 V_o k \cos(ky)$

$$\Rightarrow \sigma(0,y) = kV_0 \left[\epsilon \frac{\cosh(kd)}{\sinh(kd)} + \epsilon_0 \right] \cos(ky)$$

Όμοια

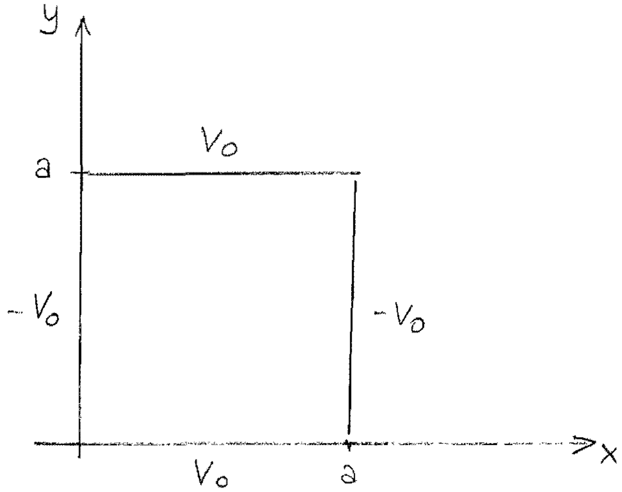
$$\sigma(x=d,y) = \hat{i}_n \circ \left(\vec{D}(x=d^-) - \vec{D}(x=d^+) \right) =$$

$$= (-\hat{i}_x) \left[\epsilon E_x(d^-)\hat{i}_x - \epsilon_0 \cdot 0 \right] =$$

$$= -\epsilon E_x(d-) =$$

$$= -\epsilon V_0 \cos(ky) k \frac{1}{\sinh(kd)}$$

ΑΣΚΗΣΗ 4:



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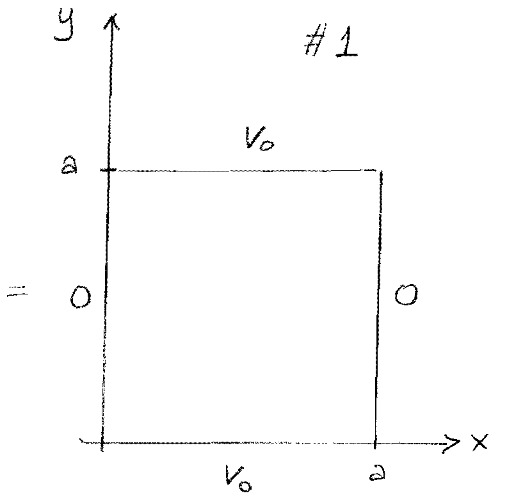

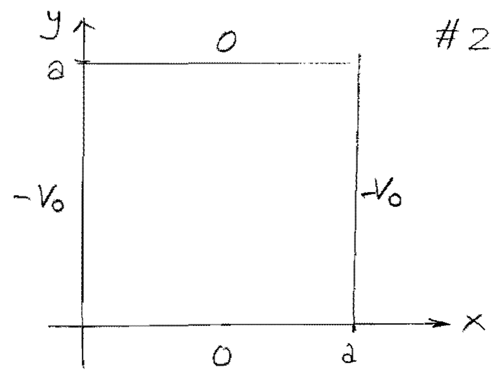

(α) χωρίζομε το πρόβλημα σε 2 υποπροβλήματα #1, & #2

Υποπρόβλημα #1:

$$X(x) = A_1 \cos(kx) + A_2 \sin(kx)$$

$$Y(y) = B_1 \cosh(ky) + B_2 \sinh(ky)$$

Όμως $X(0) = 0 \Rightarrow A_1 = 0$

$X(a) = 0 \Rightarrow A_2 \sin(ka) = 0 \Rightarrow k_n a = n\pi \quad n = 1, 2, \ldots$

$$k_n = \frac{n\pi}{a}, \quad n = 1, 2, \ldots$$

Άρα $\Phi = \sin(k_n x)\left[B_1 \cosh(k_n y) + B_2 \sinh(k_n y)\right]$

$$\Phi(x, y=0) = V_o = \sum_n \sin(k_n x)\left[B_{1n}\right]$$

$$\Phi(x, y=a) = V_o = \sum_n \sin(k_n x)\left[B_{1n}\cosh(k_n a) + B_{2n}\sinh(k_n a)\right]$$

Ορθογωνιότητα:

$$\int_0^a \sin(k_n x)\sin(k_m x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{a}{2} & n = m \end{cases}$$

$$\langle V_o, \sin k_m x \rangle = B_{1m} \frac{a}{2} \Rightarrow B_{1m} = \frac{2}{a} \int_0^a V_o \sin(k_m x) dx$$

$$\Rightarrow B_{1m} = \frac{2V_o}{a} \left[\frac{-\cos(k_m x)}{k_m} \right]_0^a \rightarrow$$

$$\rightarrow B_{1m} = \frac{2V_o}{a} \left[\frac{-\cos(k_m a) + 1}{k_m} \right] = \frac{2V_o}{k_m a} \left[1 - (-1)^m \right]$$

$$B_{1m} = \begin{cases} \dfrac{4V_o}{\frac{m\pi}{\cancel{a}} \cancel{a}} = \dfrac{4V_o}{m\pi} & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

Επίσης,

$$\langle V_o, \sin(k_m x) \rangle = \frac{a}{2} \left[B_{1m} \cosh(k_m a) + B_{2m} \sinh(k_m a) \right]$$

$$\frac{2}{a} \langle V_o, \sin(k_m x) \rangle = B_{1m} \cosh(k_m a) + B_{2m} \sinh(k_m a)$$

$$B_{2m} = \left[\frac{2}{a} \langle V_o, \sin(k_m x) \rangle - \frac{2}{a} \langle V_o, \sin(k_m x) \rangle \cosh(k_m a) \right] \frac{1}{\sinh(k_m a)}$$

$$= \frac{2V_o}{m\pi} \left[1 - (-1)^m \right] \left\{ \frac{1 - \cosh(k_m a)}{\sinh(k_m a)} \right\}$$

$$B_{2m} = \begin{cases} \dfrac{4V_o}{m\pi} \left[\dfrac{1 - \cosh(m\pi)}{\sinh(m\pi)} \right] & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

Επομένως,

$$\Phi_1(x, y) = \sum_{\substack{m = 2p+1 \\ p = 0, -}}^{\infty} \left[B_{1m} \cosh(k_m y) + B_{1m} \frac{1 - \cosh(k_m a)}{\sinh(k_m a)} \sinh(k_m y) \right] \sin k_m x$$

$$\Phi_1(x, y) = \sum_{m=2p+1} B_m \frac{1}{\sinh(k_m a)} \left\{ \sinh(k_m y) - \sinh(k_m(y-a)) \right\} \sin k_m x$$

Υποπρόβλημα #2.

Αυτό είναι ακριβώς ανάλογο αλλά με αλλαγή του $x \longleftrightarrow y$ και $V_0 \longrightarrow -V_0$.

$$X(x) = B_1 \cosh(k_n x) + B_2 \sinh(k_n x)$$

$$Y(y) = A_1 \cos(ky) + A_2 \sin(ky) \quad \sim \quad Y(y) = \sin(k_n y)$$

$$k_n = \frac{n\pi}{a}$$

Επομένως

$$\Phi_2(x,y) = \sum_{m=2p+1}^{\infty} (-B_{1m}) \left[\cosh(k_m x) + \frac{1-\cosh(k_m a)}{\sinh(k_m a)} \sinh(k_m x) \right] \sin(k_m y)$$

$$\Phi_2(x,y) = \sum_{m=2p+1} \frac{-B_m}{\sinh(k_m a)} \Big[\sinh(k_m x) - \sinh(k_m(x-a)) \Big] \sin(k_m y)$$

η συνολική λύση είναι

$$\Phi(x,y) = \Phi_1(x,y) + \Phi_2(x,y)$$

(β) $$\vec{E} = -\vec{\nabla}\Phi = -\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial x} \right) \hat{i}_x - \left(\frac{\partial \Phi_1}{\partial y} + \frac{\partial \Phi_2}{\partial y} \right) \hat{i}_y$$

$$\frac{\partial \Phi_1}{\partial x} = \sum_{\substack{m \\ m=2p+1}} \frac{B_m}{\sinh(k_m a)} \Big[\sinh(k_m y) - \sinh(k_m(y-a)) \Big] k_m \cos(k_m x)$$

$$\frac{\partial \Phi_2}{\partial x} = \sum_m \frac{-B_m}{\sinh(k_m a)} \Big[k_m \cosh(k_m x) - k_m \cosh(k_m(x-a)) \Big] \sin k_m y$$

$$\frac{\partial \Phi_1}{\partial y} = \sum_m \frac{B_m}{\sinh(k_m a)} \Big[k_m \cosh(k_m y) - k_m \cosh(k_m(y-a)) \Big] \sin k_m x$$

$$\frac{\partial \Phi_2}{\partial y} = \sum_m \frac{-B_m}{\sinh(k_m a)} \Big[\sinh(k_m x) - \sinh(k_m(x-a)) \Big] k_m \cos(k_m y)$$

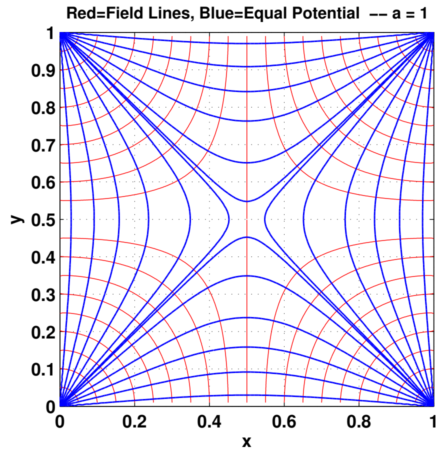

Figure 2: Electric field Lines (red) and Equipotential Lines (blue)

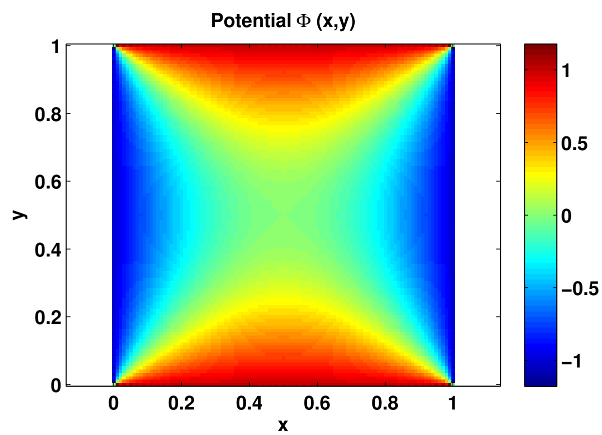

Figure 3: Colormap of the Electric Potential Distribution

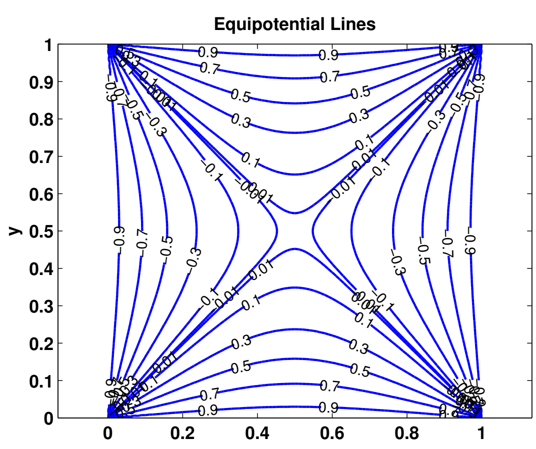

Figure 4: Equipotential Lines at Specific Potential Levels