ΣΕΙΡΑ ΑΣΚΗΣΕΩΝ No. 3

ΑΣΚΗΣΗ 1:

(α) $z(t) = 3\cos(2\pi 30 t - \pi/4) \qquad \omega = 2\pi 30$

$\rightsquigarrow \; Z = 3e^{-j\pi/4} = 3\cos(\pi/4) - j\,3\sin(\pi/4) = 3\,\dfrac{1-j}{\sqrt{2}}$

(β) $z(x,t) = 4e^{-3x}\sin(\omega t - \pi/6) =$

$= 4e^{-3x}\cos\left(\omega t - \frac{\pi}{6} - \frac{\pi}{2}\right) = 4e^{-3x}\cos\left(\omega t - \frac{2\pi}{3}\right)$

$\rightsquigarrow \; Z = 4e^{-3x}e^{-j2\pi/3}$

(γ) $z(t) = 2\sin[\omega t + \pi/3] + 3\cos[\omega t - \pi/6] \Rightarrow$

$z(t) = 2\cos\left[\omega t + \frac{\pi}{3} - \frac{\pi}{2}\right] + 3\cos[\omega t - \pi/6] =$

$= 2\cos[\omega t - \pi/6] + 3\cos[\omega t - \pi/6] = 5\cos(\omega t - \pi/6)$

$\rightsquigarrow \; Z = 5e^{-j\pi/6}$

(δ) $Z = \sqrt{j}\,6\,e^{j\pi/4} = \left(e^{j\pi/2}\right)^{1/2} 6\,e^{j\pi/4} = 6e^{j\pi/4}e^{j\pi/4} = 6e^{j\pi/2}$

$z(t) = \mathrm{Re}\{Z e^{j\omega t}\} = \mathrm{Re}\{6e^{j\pi/2}e^{j\omega t}\} =$

$= 6\cos(\omega t + \pi/2) = -6\sin(\omega t)$

(ε) $Z = 3 - j4 = \sqrt{3^2 + 4^2}\, e^{-j\tan^{-1}(4/3)} = 5e^{-j53.13°}$

$z(t) = \mathrm{Re}\{Z e^{j\omega t}\} = \mathrm{Re}\{5e^{-j53.13°}e^{j\omega t}\} =$

$= 5\cos(\omega t - 53.13°)$

(στ) $Z = -3e^{j\pi/3} = 3e^{j\pi + j\pi/3} = 3e^{j4\pi/3}$

$z(t) = \mathrm{Re}\{3e^{j4\pi/3}e^{j\omega t}\} = 3\cos(\omega t + 4\pi/3)$

ΑΣΚΗΣΗ 2:

$$\vec{E} = E_o\,(\hat{\imath}_x + \hat{\imath}_y - \hat{\imath}_z)\, e^{-j\frac{2\omega}{c}\left(\frac{x}{\sqrt6}+\frac{y}{\sqrt6}+\frac{2z}{\sqrt6}\right)}$$

(α) $\vec{k}\cdot\vec{r} = \frac{2\omega}{c}\left(\frac{x}{\sqrt6}+\frac{y}{\sqrt6}+\frac{2z}{\sqrt6}\right) =$

$$= \underbrace{\frac{2\omega}{c}\left(\frac{1}{\sqrt6}\hat{\imath}_x + \frac{1}{\sqrt6}\hat{\imath}_y + \frac{2}{\sqrt6}\hat{\imath}_z\right)}_{\vec{k}} \cdot \underbrace{\left(x\hat{\imath}_x + y\hat{\imath}_y + z\hat{\imath}_z\right)}_{\vec{r}}$$

Επομένως $\vec{k} = \frac{2\omega}{c}\frac{1}{\sqrt6}(\hat{\imath}_x + \hat{\imath}_y + 2\hat{\imath}_z) = k\,\hat{k} =$

$$= \frac{2\omega}{c}\underbrace{\left[\frac{1}{\sqrt6}(\hat{\imath}_x + \hat{\imath}_y + 2\hat{\imath}_z)\right]}_{\hat{k}} \rightsquigarrow k = \frac{2\omega}{c}$$

Όμως $\left.\begin{aligned} k &= k_o\sqrt{\varepsilon_r\mu_r} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r} \\ k &= \frac{2\omega}{c}\end{aligned}\right\} \Longrightarrow \sqrt{\mu_r\varepsilon_r} = 2$

$\mu_r = 1$ (μη μαγνητικό υλικό)

οπότε $\varepsilon_r = 4$

(β) Το κυματοδιάνυσμα ήδη βρέθηκε $\vec{k} = \frac{2\omega}{c}\frac{1}{\sqrt6}(\hat{\imath}_x + \hat{\imath}_y + 2\hat{\imath}_z)$

παρατηρούμε ότι $\vec{E}_o = E_o(\hat{\imath}_x + \hat{\imath}_y - \hat{\imath}_z)$ και

$\vec{k}\cdot\vec{E}_o = E_o\left(\frac{2\omega}{c}\right)\frac{1}{\sqrt6}(1+1-2) = 0$ όπως είναι αναγκαίο από τις εξισώσεις του Maxwell.

(γ) $Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_r\varepsilon_o}} = \frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{Z_o}{\sqrt4} = \frac{Z_o}{2} \simeq 60\pi\ (\Omega)$

(δ) $\vec{P} = \frac{1}{2}\mathrm{Re}\{\vec{E}\times\vec{H}^*\}$

$$\vec{H} = \frac{1}{Z}(\hat{k}\times\vec{E}) = \frac{1}{Z}\begin{vmatrix} \hat{\imath}_x & \hat{\imath}_y & \hat{\imath}_z \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{2}{\sqrt6} \\ E_oA & E_oA & -E_oA \end{vmatrix} \Longrightarrow$$

(όπου $A = e^{-j\frac{2\omega}{c}\left(\frac{x}{\sqrt6}+\frac{y}{\sqrt6}+\frac{2z}{\sqrt6}\right)}$)

$$\vec{H} = \frac{1}{Z}\left[\hat{\imath}_x\left(-\frac{3E_o}{\sqrt6}\right)A - \hat{\imath}_y\left(-\frac{3E_o}{\sqrt6}\right)A + \hat{\imath}_z(\emptyset)A\right] =$$

$$= \frac{E_o}{Z}\frac{3}{\sqrt6}(-\hat{\imath}_x + \hat{\imath}_y)\, e^{-j\frac{2\omega}{c}\left(\frac{x}{\sqrt6}+\frac{y}{\sqrt6}+\frac{2z}{\sqrt6}\right)}$$

$$\vec{H}^* = \frac{E_o}{Z}\frac{3}{\sqrt{6}}\left(-\hat{i}_x+\hat{i}_y\right)A^* \quad = H_x^*\hat{i}_x + H_y^*\hat{i}_y$$

Επομένως:

$$\vec{P} = \frac{1}{2}\,\mathrm{Re}\left\{\begin{vmatrix}\hat{i}_x & \hat{i}_y & \hat{i}_z \\ E_oA & E_oA & -E_oA \\ H_x^* & H_y^* & 0\end{vmatrix}\right\} =$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{\hat{i}_x\left(E_oA\,\frac{E_o}{Z}\frac{3}{\sqrt{6}}A^*\right) - \hat{i}_y\left(E_oA\left(-\frac{E_o}{Z}\right)\frac{3}{\sqrt{6}}A^*\right) + \hat{i}_z\left(E_oA\,\frac{E_o}{Z}\frac{3}{\sqrt{6}}A^*\right.\right.$$

$$\left.\left. - E_oA\left(-\frac{E_o}{Z}\frac{3}{\sqrt{6}}A^*\right)\right)\right\} \Rightarrow \qquad (AA^*=1)$$

$$\vec{P} = \frac{1}{2}\,\mathrm{Re}\left\{\frac{E_o^2}{Z}\frac{3}{\sqrt{6}}\hat{i}_x + \frac{E_o^2}{Z}\frac{3}{\sqrt{6}}\hat{i}_y + 2\,\frac{E_o^2}{Z}\frac{3}{\sqrt{6}}\hat{i}_z\right\} =$$

$$= \frac{1}{2}\,\frac{E_o^2}{Z}\frac{3}{\sqrt{6}}\left(\hat{i}_x+\hat{i}_y+2\hat{i}_z\right) = \frac{E_o^2}{120\pi}\frac{3}{\sqrt{6}}\left(\hat{i}_x+\hat{i}_y+2\hat{i}_z\right) \qquad (W/m^2)$$

ΑΣΚΗΣΗ 3:

$$\vec{H}(y=0,t) = \hat{i}_x\, 0.1 \sin(10^{10}\pi t - \pi/3) \quad (A/m)$$

$\omega = 10^{10}\pi$ (rad/sec) $\qquad \varepsilon_r = 80,\ \mu_r = 1,\ \sigma = 4$ S/m

(α)
$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left(-1 + \left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{1/2}\right)\right\}^{1/2} = \omega\sqrt{\mu\epsilon}\ A$$

$$\beta = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left(+1 + \left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{1/2}\right)\right\}^{1/2} = \omega\sqrt{\mu\epsilon}\ B$$

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{10^{10}\pi\ 8,854\cdot 10^{-12}\ 80} = 0.17975$$

$$A = \left\{\frac{1}{2}\left[-1 + \left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{1/2}\right]\right\}^{1/2} = 8.95194477\cdot 10^{-2}$$

$$B = \left\{\frac{1}{2}\left[+1 + \left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{1/2}\right]\right\}^{1/2} = 1.00399887$$

$$\alpha = \omega\sqrt{\mu\epsilon}\ A = 10^{10}\pi\left[4\pi\cdot 10^{-7}\ 80\cdot 8.854\ 10^{-12}\right]^{1/2} 8,95194477\cdot 10^{-2}\ 1/m$$
$$= 10^{10}\pi\ 29.8345\ 10^{-9}\ 8,95194477\cdot 10^{-2}\ (1/m) = 83.9\ (1/m)$$

$$\beta = \omega\sqrt{\mu\epsilon}\ B = 10^{10}\pi\left[29.8345\cdot 10^{-9}\right] 1.00399887 =$$
$$= 941.03\ (1/m)$$

$$Z = \left[\frac{j\omega\mu}{\sigma + j\omega\epsilon}\right]^{1/2} = \left[\frac{j\omega\mu}{j\omega\epsilon\left[1+\frac{\sigma}{j\omega\epsilon}\right]}\right]^{1/2} = \frac{\sqrt{\mu/\epsilon}}{\left[1-j\frac{\sigma}{\omega\epsilon}\right]^{1/2}} =$$

$$= \frac{\sqrt{\mu_o/\epsilon_o}\ (1/\sqrt{\varepsilon_r})}{\left(1-j\frac{\sigma}{\omega\epsilon}\right)^{1/2}} = \frac{376.734\ (1/\sqrt{80})}{(1-j\,0.17975)^{1/2}} = 41.786\ \underline{|+5.0952^\circ}\ (\Omega)$$

$$\rightarrow Z = 41.786\ \underline{|+5.0952^\circ}\ (\Omega) = 41.62 + j\,3.71\ (\Omega)$$

$$u = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{80}} = 1,11803399\ 10^{-1}c = 3.354\cdot 10^{7}\ m/s$$

(χρησιμοποιώντας $c \simeq 3\cdot 10^8$ m/s)

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{941.03}\ m = 6.677\cdot 10^{-3}\ m = 6.677\ mm$$

$$\delta = 1/\alpha = (1/83.9)\ m = 1,1919\ 10^{-2}\ m = 1,1919\ cm$$

(β) $\vec{H}(y,t) = \hat{i}_x\, 0.1\, e^{-\alpha y} \cos[10^{10}\pi t - \frac{\pi}{3} - \frac{\pi}{2} - \beta y]$

$|\vec{H}| = 0.01 \Rightarrow \quad 0.01 = 0.1\, e^{-\alpha y} \Rightarrow \quad e^{-\alpha y} = 0.1 \Rightarrow$

$\Rightarrow -\alpha y = \ln(0.1) \Rightarrow \quad y = -\frac{\ln(0.1)}{\alpha} = 2.7444 \cdot 10^{-2}\, m$

(γ) $\vec{H}(y=0.5m, t) = \hat{i}_x\, 0.1\, e^{-\alpha 0.5} \cos[10^{10}\pi t - \frac{\pi}{3} - \frac{\pi}{2} - 941.03 \cdot 0.5]$

$$= \hat{i}_x\, 6.044 \cdot 10^{-20} \cos[10^{10}\pi t - \frac{\pi}{3} - \frac{\pi}{2} - 74.884(2\pi)]$$

$$= \hat{i}_x\, 6.044 \cdot 10^{-20} \cos[10^{10}\pi t - \frac{\pi}{3} - \frac{\pi}{2} - 0.884 \cdot 2\pi]$$

$$= \hat{i}_x\, 6.044 \cdot 10^{-20} \cos[10^{10}\pi t - 2.6029\pi]$$

$$= \hat{i}_x\, 6.044 \cdot 10^{-20} \cos[10^{10}\pi t - 0.6029\pi]$$

Είναι ευκολότερο να βρούμε τον φασιθέτη του $\vec{E}$ και μετά το στιγμιαίο πεδίο. Ο φασιθέτης του μαγνητικού πεδίου είναι:

$$\vec{H} = \hat{i}_x\, 0.1 e^{-\alpha y}\, e^{-j\beta y}\, e^{-j5\pi/6}$$

Από τις εξισώσεις του Maxwell: (για επίπεδο κύμα):

$$-j(\vec{k} \times \vec{H}) = (\sigma + j\omega\epsilon)\vec{E} \quad \Rightarrow \quad \vec{E} = \frac{-j}{\sigma + j\omega\epsilon} k_c (\hat{k} \times \vec{H}) \Rightarrow$$

$$\vec{E} = -\frac{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}}{(\sigma + j\omega\epsilon)} (\hat{k} \times \vec{H}) = -\left(\frac{j\omega\mu}{\sigma + j\omega\epsilon}\right)^{1/2} \hat{k} \times \vec{H} =$$

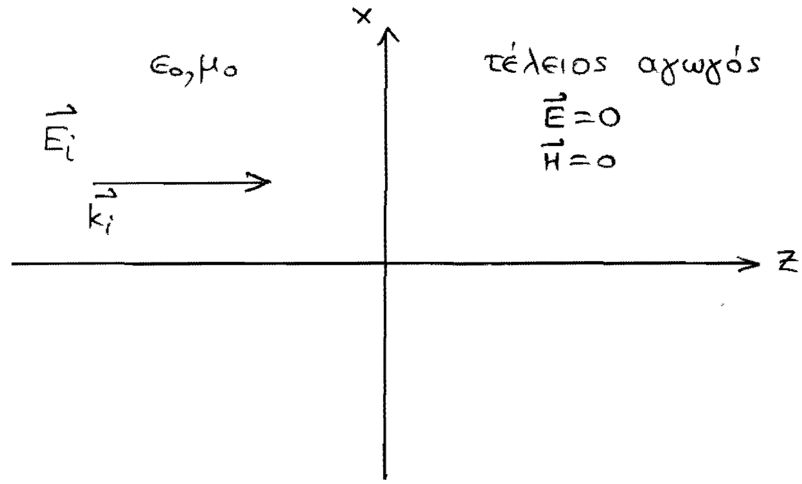
$$= -Z(\hat{k} \times \vec{H}) = -Z[\hat{i}_y \times \hat{i}_x H] =$$

$$= -ZH(-\hat{i}_z) = \hat{i}_z\, 41.786\, e^{j5.0952°}\, 0.1\, e^{-\alpha y} e^{-j\beta y} e^{-j5\pi/6}$$

$$\Rightarrow \vec{E}(y=0.5m, t) = \hat{i}_z\, 41.786 \cdot 0.1 \cdot e^{-\alpha 0.5} \cos[10^{10}\pi t - \frac{5\pi}{6} - 941.03\, 0.5 + 5.0952°]$$

$$= \hat{i}_z\, 2.526 \cdot 10^{-18} \cos[10^{10}\pi t - 0.6029\pi + \frac{5.0952 \times \pi}{180}]$$

ΑΣΚΗΣΗ 4:



$$\vec{E_i} = E_0 [2\hat{i}_x - j\hat{i}_y] e^{-j\beta z}$$

(α) $\vec{k_i} = k_0 \hat{i}_z \qquad k_0 = \omega\sqrt{\mu_0\epsilon_0}$

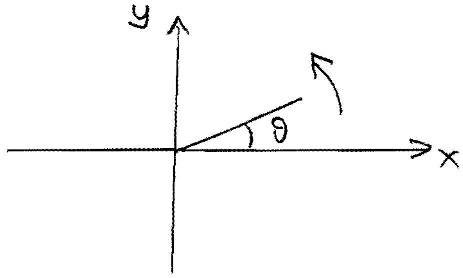
$\vec{k_i}\cdot\vec{r} = k_0 z = \beta z \sim \beta = \omega\sqrt{\mu_0\epsilon_0}$

$E_x = 2E_0 e^{-j\beta z} \sim \mathcal{E}_x = 2E_0 \cos(\omega t - \beta z)$

$E_y = -jE_0 e^{-j\beta z} \sim \mathcal{E}_y = E_0 \cos(\omega t - \beta z - \frac{\pi}{2}) =$

$= E_0 \sin(\omega t - \beta z)$

$\left(\frac{\mathcal{E}_x}{2E_0}\right)^2 + \left(\frac{\mathcal{E}_y}{E_0}\right)^2 = \cos^2(\ ) + \sin^2(\ ) = 1 \sim$ ελλειπτική πόλωση



$\phi = \phi_y - \phi_x = -\pi/2$

$\tan\theta = \frac{\mathcal{E}_y}{\mathcal{E}_x} = \frac{1}{2}\tan(\omega t - \beta z)$

RH ελλειπτική πόλωση

(β) $\vec{E}_1 = \vec{E}_i + \vec{E}_r$

$\vec{E}_r = (E_{rx}\hat{i}_x + E_{ry}\hat{i}_y)\, e^{+jk_0 z}$

Εφαπτομενικές συνιστώσες του ηλεκτρικού πεδίου συνεχείς:

$E_{x1} = E_{x2} = 0 \sim 2E_0 + E_{rx} = 0 \sim E_{rx} = -2E_0$

$E_{y1} = E_{y2} = 0 \sim -jE_0 + E_{ry} = 0 \sim E_{ry} = +jE_0$

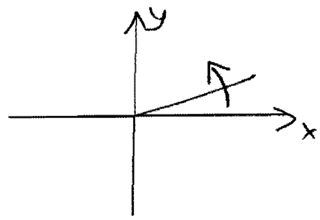
$$\vec{E}_r = (-2E_0\,\hat{\imath}_x + jE_0\,\hat{\imath}_y)\,e^{+jk_0z}$$

$$\mathcal{E}_{rx} = 2E_0\cos(\omega t + kz + \pi) = -2E_0\cos(\omega t + k_0 z)$$

$$\mathcal{E}_{ry} = E_0\cos(\omega t + kz + \pi/2) = -E_0\sin(\omega t + k_0 z)$$

$$\left(\frac{\mathcal{E}_{rx}}{2E_0}\right)^2 + \left(\frac{\mathcal{E}_{ry}}{E_0}\right)^2 = 1 \sim \text{ελλειπτική πόλωση}$$

$$\tan\vartheta = \frac{\mathcal{E}_{ry}}{\mathcal{E}_{rx}} = \frac{1}{2}\tan(\omega t + k_0 z)$$



διάδοση $-z$

οπότε είναι LH ελλειπτική πόλωση

$$\vec{H}_r = \frac{1}{Z_0}(-\hat{\imath}_z \times \vec{E}_r) = \frac{-1}{Z_0}\,\hat{\imath}_z \times (-2E_0\,\hat{\imath}_x + jE_0\,\hat{\imath}_y)\,e^{jk_0z}$$

$$= \left(\frac{2E_0}{Z_0}\hat{\imath}_y + j\frac{E_0}{Z_0}\hat{\imath}_x\right)e^{jk_0z}$$

$$\vec{H}_i = \frac{1}{Z_0}(+\hat{\imath}_z \times \vec{E}_i) = \frac{1}{Z_0}\,\hat{\imath}_z \times (2E_0\,\hat{\imath}_x - jE_0\,\hat{\imath}_y)\,e^{-jk_0z} =$$

$$= \left(\frac{2E_0}{Z_0}\hat{\imath}_y + j\frac{E_0}{Z_0}\hat{\imath}_x\right)e^{-jk_0z}$$

(γ) $$\vec{E}_1 = \vec{E}_i + \vec{E}_r = E_0[2\hat{\imath}_x - j\hat{\imath}_y)\,e^{-jk_0z} + E_0[-2\hat{\imath}_x + j\hat{\imath}_y]e^{+jk_0z}$$

$$= 2E_0\,(e^{-jk_0z} - e^{+jk_0z})\,\hat{\imath}_x + E_0 j\,[e^{jk_0z} - e^{-jk_0z}]\,\hat{\imath}_y =$$

$$= 2E_0\,(-2j\sin(k_0z))\,\hat{\imath}_x + E_0 j\, 2j\sin(k_0z)\,\hat{\imath}_y$$

$$\vec{\mathcal{E}}_1 = \hat{\imath}_x\, 4E_0\sin(k_0z)\cos(\omega t - \tfrac{\pi}{2})\,\hat{\imath}_x - \hat{\imath}_y\, 2E_0\sin(k_0z)\cos(\omega t) =$$

$$= \hat{\imath}_x\, 4E_0\sin(k_0z)\sin(\omega t) - \hat{\imath}_y\, 2E_0\sin(k_0z)\cos(\omega t)$$

$$= E_0\sin(k_0z)\,[4\hat{\imath}_x\sin(\omega t) - 2\hat{\imath}_y\cos(\omega t)]$$

(δ) $\sigma = \hat{i}_n \cdot (\vec{D}^+ - \cancel{\vec{D}^-}^{\,0}) =$

$$= (-\hat{i}_z) \cdot (\epsilon_0 \vec{E}_1) = 0 \sim \sigma = 0$$

$$\vec{K} = \hat{i}_n \times (\vec{H}^+ - \cancel{\vec{H}^-}^{\,0}) = (-\hat{i}_z) \times \{\vec{H}_i + \vec{H}_r\}\Big|_{z=0} =$$

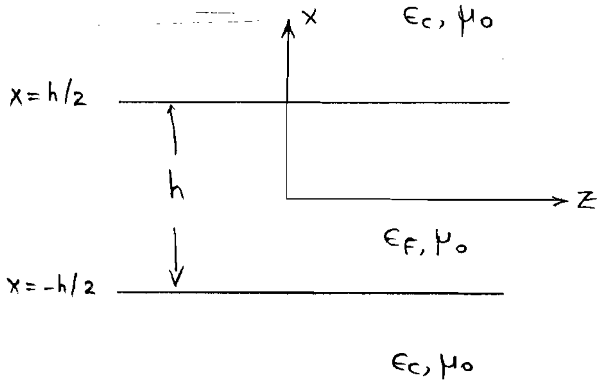
$$= (-\hat{i}_z) \times \left[\frac{4E_0}{Z_0} \hat{i}_y + j \frac{2E_0}{Z_0} \hat{i}_x \right] =$$

$$= \hat{i}_x \frac{4E_0}{Z_0} - \hat{i}_y \frac{2E_0}{Z_0} j$$

$$\vec{\mathcal{K}} = \mathrm{Re}\{\vec{K} e^{j\omega t}\} = \frac{4E_0}{Z_0} \cos(\omega t)\, \hat{i}_x + \frac{2E_0}{Z_0} \cos(\omega t - \tfrac{\pi}{2})\, \hat{i}_y$$

$$= \frac{2E_0}{Z_0} \left[2\cos(\omega t)\, \hat{i}_x + \sin(\omega t)\, \hat{i}_y \right]$$

ΑΣΚΗΣΗ 5:



$$\vec{E} = \hat{i}_y \begin{cases} E_c\, e^{-\gamma_c (x-h/2)} & h/2 < x < \infty \\ E_f \cos(k_f x) & -\frac{h}{2} < x < \frac{h}{2} \\ E_c\, e^{\gamma_c (x+h/2)} & -\infty < x < -h/2 \end{cases} e^{-j\beta z}$$

(α) Από τις εξισώσεις Maxwell στην μόνιμη ημιτονοειδή κατάσταση έχουμε:

$$\vec{\nabla}\times\vec{E} = -j\omega\mu_o \vec{H} \Rightarrow \vec{H} = -\frac{1}{j\omega\mu_o}\vec{\nabla}\times\vec{E} \Rightarrow$$

$$\vec{H} = -\frac{1}{j\omega\mu_o}\begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{1}{j\omega\mu_o}\left[\hat{i}_x\left(-\frac{\partial E_y}{\partial z}\right) + \hat{i}_z\left(\frac{\partial E_y}{\partial x}\right)\right]$$

Επομένως

$$H_x = \frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial z} = \frac{1}{j\omega\mu_o}(-j\beta)\begin{Bmatrix} E_c\, e^{-\gamma_c(x-h/2)} \\ E_f \cos(k_f x) \\ E_c\, e^{+\gamma_c(x+h/2)} \end{Bmatrix} e^{-j\beta z}$$

$$= -\frac{\beta}{\omega\mu_o} e^{-j\beta z}\begin{cases} E_c\, e^{-\gamma_c(x-h/2)} & h/2 < x < \infty \\ E_f \cos(k_f x) & -h/2 < x < h/2 \\ E_c\, e^{+\gamma_c(x+h/2)} & -\infty < x < -h/2 \end{cases}$$

$$H_z = -\frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial x} = -\frac{1}{j\omega\mu_o} e^{-j\beta z}\begin{cases} E_c(-\gamma_c)\, e^{-\gamma_c(x-h/2)} \\ E_f(-k_f)\sin(k_f x) \\ E_c(+\gamma_c)\, e^{\gamma_c(x+h/2)} \end{cases}$$

$$= \frac{1}{j\omega\mu_o} e^{-j\beta z}\begin{cases} \gamma_c E_c\, e^{-\gamma_c(x-h/2)} & h/2 < x < \infty \\ k_f E_f \sin(k_f x) & -h/2 < x < h/2 \\ -\gamma_c E_c\, e^{\gamma_c(x+h/2)} & -\infty < x < -h/2 \end{cases}$$

Επομένως $\vec{H} = \vec{H}_x \hat{i}_x + H_z \hat{i}_z$

(β) Εφόσον τα πεδία είναι άθροισμα επιπέδων κυμάτων ικανοποιούν τις εξισώσεις Maxwell. Αρκεί λοιπόν να ικανοποιούν και τις οριακές συνθήκες.

$\underline{x = h/2}$:

$$\left.\begin{aligned} E_y(x = h/2^+) - E_y(x = h/2^-) &= 0 \\ H_z(x = h/2+) - H_z(x = h/2-) &= 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} E_c &= E_f \cos(k_f h/2) \\ \gamma_c E_c &= k_f E_f \sin(k_f h/2) \end{aligned}\right.$$

$\underline{x = -h/2}$:

$$\left.\begin{aligned} E_y(x = -h/2+) - E_y(x = -h/2^-) &= 0 \\ H_z(x = -h/2+) - H_z(x = -h/2^-) &= 0 \end{aligned}\right\} \Rightarrow$$

$$E_f \cos(k_f(-h/2)) = E_c$$

$$k_f E_f \sin\left(k_f\left(-\frac{h}{2}\right)\right) = -\gamma_c E_c$$

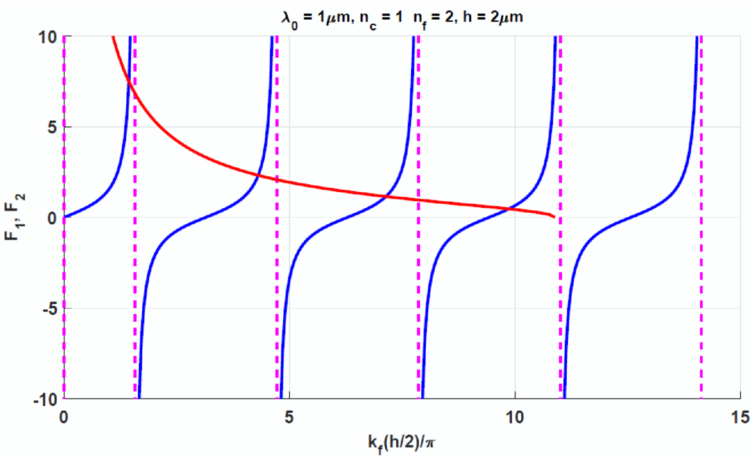
Οπότε για να ικανοποιούνται και οι οριακές συνθήκες πρέπει να ισχύουν οι σχέσεις:

$$\left.\begin{aligned} E_c &= E_f \cos(k_f h/2) \\ \gamma_c E_c &= k_f E_f \sin(k_f h/2) \end{aligned}\right\} \Rightarrow$$

$$\gamma_c = k_f \tan\left(k_f \frac{h}{2}\right) \Rightarrow \tan\left(k_f \frac{h}{2}\right) = \frac{\gamma_c}{k_f}$$

Όμως $\gamma_c = (\beta^2 - k_0^2 n_c^2)^{1/2}$, $k_f = (k_0^2 n_f^2 - \beta^2)^{1/2}$. Είναι πιθανές πολλές λύσεις.

παράδειγμα:
4 ρυθμοί φαίνεται ότι υπάρχουν για τα δεδομένα του παραδείγματος



(γ) $\vec{\mathcal{E}} = \mathrm{Re}\{\vec{E}\, e^{j\omega t}\} \Rightarrow$

$$\vec{\mathcal{E}}(x,z,t) = \hat{\imath}_y \begin{cases} E_c\, e^{-\gamma_c(x-h/2)} \cos(\omega t-\beta z) & h/2 < x < \infty \\ E_f \cos(k_f x)\cos(\omega t-\beta z) & -h/2 < x < h/2 \\ E_c\, e^{\gamma_c(x+h/2)} \cos(\omega t-\beta z) & -h/2 < x < \infty \end{cases}$$

και $E_c = E_f \cos(k_f h/2)$

(δ) Το μιγαδικό διάνυσμα Poynting δίδεται από την σχέση:

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* \Rightarrow$$

$$\vec{S} = \frac{1}{2}E_y\hat{\imath}_y \times (H_x^*\hat{\imath}_x + H_z^*\hat{\imath}_z) =$$

$$= \frac{1}{2}E_y H_x^*(-\hat{\imath}_z) + \frac{1}{2}E_y H_z^*\hat{\imath}_x = S_x\hat{\imath}_x + S_z\hat{\imath}_z$$

$\frac{h}{2} < x < \infty$:

$$S_z = -\frac{1}{2}E_c e^{-\gamma_c(x-h/2)} e^{-j\beta z}\left(-\frac{\beta}{\omega\mu_0}\right)E_c^* e^{-\gamma_c(x-h/2)} e^{+j\beta z}$$

$$= \frac{1}{2}|E_c|^2 e^{-2\gamma_c(x-h/2)}\left(\frac{\beta}{\omega\mu_0}\right)$$

$$S_x = \frac{1}{2}E_c e^{-\gamma_c(x-h/2)} e^{-j\beta z}\frac{1}{-j\omega\mu_0} e^{+j\beta z}\gamma_c E_c^* e^{-\gamma_c(x-h/2)}$$

$$= \frac{1}{2}|E_c|^2\frac{j\gamma_c}{\omega\mu_0} e^{-2\gamma_c(x-h/2)}$$

Επομένως $\langle N_z\rangle = \mathrm{Re}\{S_z\} = \frac{1}{2}|E_c|^2 e^{-2\gamma_c(x-h/2)}\left(\frac{\beta}{\omega\mu_0}\right)$

$\langle N_x\rangle = 0$

$-h/2 < x < h/2$:

$$S_z = -\frac{1}{2}E_f\cos(k_f x)e^{-j\beta z}\left(-\frac{\beta}{\omega\mu_0}\right)e^{+j\beta z}E_f^*\cos(k_f x) =$$

$$= \frac{1}{2}|E_f|^2\cos^2(k_f x)\left(\frac{\beta}{\omega\mu_0}\right)$$

$$S_x = \frac{1}{2}E_f\cos(k_f x)e^{-j\beta z}\frac{1}{-j\omega\mu_0}e^{+j\beta z}E_f^* k_f\sin(k_f x) =$$

$$= \frac{1}{2}|E_f|^2\frac{jk_f}{\omega\mu_0}\sin(k_f x)\cos(k_f x)$$

$\langle N_z\rangle = \mathrm{Re}\{S_z\} = \frac{1}{2}|E_f|^2\cos^2(k_f x)\left(\frac{\beta}{\omega\mu_0}\right)$, $\langle N_x\rangle = 0$

$-\infty < x < -h/2$:

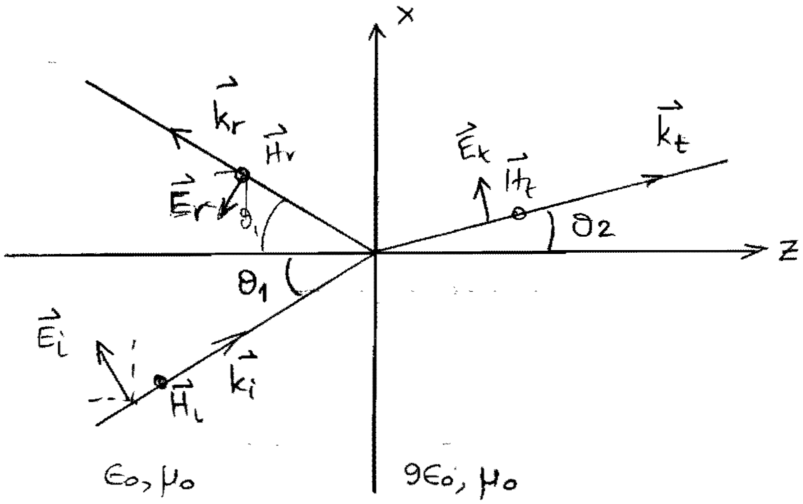
$$S_z = -\frac{1}{2} E_c e^{\gamma_c(x+h/2)} e^{-j\beta z} \left(-\frac{\beta}{\omega\mu_0}\right) E_c^* e^{+\gamma_c(x+\frac{h}{2})} e^{+j\beta z}$$
$$= \frac{1}{2} |E_c|^2 \left(\frac{\beta}{\omega\mu_0}\right) e^{2\gamma_c(x+h/2)}$$

$$S_x = \frac{1}{2} E_c e^{\gamma_c(x+h/2)} e^{-j\beta z} \left(\frac{1}{-j\omega\mu_0}\right) e^{+j\beta z} (-\gamma_c E_c^*) e^{\gamma_c(x+h/2)}$$
$$= \frac{1}{2} |E_c|^2 \left(\frac{-j\gamma_c}{\omega\mu_0}\right) e^{2\gamma_c(x+h/2)}$$

$$\langle N_z \rangle = \mathrm{Re}\{S_z\} = \frac{1}{2} |E_c|^2 \left(\frac{\beta}{\omega\mu_0}\right) e^{2\gamma_c(x+h/2)}$$

$$\langle N_x \rangle = \mathrm{Re}\{S_x\} = 0$$

ΑΣΚΗΣΗ 6:



(α) $\vec{H_i} = 100\,\hat{i}_y\, e^{-j(50x+50\sqrt{3}z)}$

$k_x = 50\ m^{-1}$ $\quad k_z = 50\sqrt{3}\ m^{-1}$ $\qquad k_x = k_o \sin\vartheta_1 \quad k_z = k_o\cos\vartheta_1$

$k_i = (k_x^2 + k_z^2)^{1/2} = [50^2 + (50\sqrt{3})^2]^{1/2} = 50\cdot 2 = 100\,m^{-1}$

$k_i = k_o = \frac{\omega}{c} = \frac{2\pi}{\lambda_o} \rightarrow \lambda_o = \frac{2\pi}{k_i} = 0.0628\,m^{-1} = 6.28\,cm$

$\omega = k_o c = 100\frac{1}{m}\, 300\,000\cdot 10^3\,\frac{m}{s} = 3\cdot 10^{10}\ rad/sec \Rightarrow$

$f = \frac{\omega}{2\pi} = 4.7746\cdot 10^9\,Hz = 4.7746\ GHz$

$\vartheta_1 = \tan^{-1}(k_x/k_z) = \tan^{-1}(1/\sqrt{3}) = 30^\circ$

Η πόλωση είναι γραμμική αφού $\vec{H} \,//\, \hat{i}_y$ και συγκεκριμένα TM (//).

(β) $\vec{E_i} = -Z_1(\vec{k_i}\times\vec{H_i}) = -Z_1 \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \sin\vartheta_1 & 0 & \cos\vartheta_1 \\ 0 & H_i & 0 \end{vmatrix} e^{-j\vec{k_i}\cdot\vec{r}}$

$= -Z_1 e^{-j\vec{k_i}\cdot\vec{r}} \left[\hat{i}_x(-H_i\cos\vartheta_1) - \hat{i}_y\, 0 + \hat{i}_z H_i\sin\vartheta_i\right]$

$= (Z_1 H_i) e^{-j\vec{k_i}\cdot\vec{r}} \left[\cos\vartheta_1\,\hat{i}_x - \hat{i}_z\sin\vartheta_1\right] =$

$= E_i\left[\cos\vartheta_1\,\hat{i}_x - \hat{i}_z\sin\vartheta_1\right] e^{-j(50x+50\sqrt{3}z)}$

$$E_r = r_{TM} E_i$$

$$r_{TM} = \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \rightsquigarrow$$

$$\theta_2 = \sin^{-1}\left(\frac{n_1 \sin\theta_1}{n_2}\right)$$

$$n_1 = 1.0, \quad n_2 = \sqrt{9} = 3$$

$$\theta_2 = 9.59°$$

$$r_{TM} = 0.44978$$

$$t_{TM} = \frac{2 Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} = 0.48326$$

Επομένως,

$$\vec{E}_r = 0.44978 \cdot 377 \cdot 100 \; e^{-j \vec{k}_r \cdot \vec{r}} \left[-\cos\theta_1 \hat{\imath}_x - \sin\theta_1 \hat{\imath}_z \right]$$

$$\vec{k}_r = k_0 \left[-\hat{\imath}_z \cos\theta_1 + \sin\theta_1 \hat{\imath}_x \right]$$

$$\vec{E}_r = 16956{,}7 \left(\frac{V}{m}\right) \left[-\frac{\sqrt{3}}{2} \hat{\imath}_x - \frac{1}{2} \hat{\imath}_z \right] e^{-j(50x - 50\sqrt{3} z)}$$

Η πόλωση του ανακλώμενου κύματος παραμένει γραμμική (TM).

(γ) $E_t = t_{TM} E_i = 0{,}48326 \cdot 377 \cdot 100 = 18218{,}9 \; (V/m)$

$$\vec{E}_t = 18218{,}9 \left[-\sin\theta_2 \hat{\imath}_z + \cos\theta_2 \hat{\imath}_x \right] e^{-j \vec{k}_t \cdot \vec{r}}$$

$$\vec{k}_t = k_0 n_2 \left[\cos\theta_2 \hat{\imath}_z + \sin\theta_2 \hat{\imath}_x \right] =$$

$$= 300 \left[0.986 \hat{\imath}_z + 0{,}16667 \hat{\imath}_x \right]$$

$$= 295.8 \hat{\imath}_z + 50 \hat{\imath}_x$$

$$\vec{E}_t = 18218{,}9 \left[-\sin\theta_2 \hat{\imath}_z + \cos\theta_2 \hat{\imath}_x \right] e^{-j(295.8z + 50x)}$$

(δ) $$\vec{H}_t = \frac{1}{Z_2} \hat{k}_t \times \vec{E}_t = \frac{E_t}{Z_2} \begin{vmatrix} \hat{\imath}_x & \hat{\imath}_y & \hat{\imath}_z \\ \sin\theta_2 & 0 & \cos\theta_2 \\ \cos\theta_2 & 0 & -\sin\theta_2 \end{vmatrix} e^{-j \vec{k}_t \cdot \vec{r}}$$

$$Z_2 = \frac{377}{3} = 125.67 \, \Omega$$

$$\vec{H}_t = 144{,}98\left[\hat{\imath}_x \varnothing - \hat{\imath}_y(-\sin^2\theta_2 - \cos^2\theta_2) + \hat{\imath}_z \varnothing\right] e^{-j\vec{k}_t \cdot \vec{r}}$$

$$= 144{,}98\,\hat{\imath}_y\, e^{-j(295{,}8z + 50x)}$$

$$\vec{\mathcal{H}}_t = 144{,}98\,\hat{\imath}_y \cos\left[2\pi f t - (295{,}8z + 50x)\right]$$

$$f = 4{,}7746 \cdot 10^9\ \text{Hz}$$

(ε) $$\vec{\mathcal{P}}_{avg\,2} = \frac{1}{2}\,\text{Re}\{\vec{E}_t \times \vec{H}_t^*\} =$$

$$= \frac{1}{2}\,\text{Re}\left\{-Z_2(\hat{k}_t \times \vec{H}_t) \times \vec{H}_t^*\right\} =$$

$$= \frac{Z_2}{2}\,\text{Re}\left\{\vec{H}_t^* \times (\hat{k}_t \times \vec{H}_t)\right\} =$$

$$= \frac{Z_2}{2}\,\text{Re}\left\{\hat{k}_t |\vec{H}_t|^2 - \vec{H}_t\,(\underbrace{\hat{k}_t \cdot \vec{H}_t^*}_{0})\right\} =$$

$$= \frac{1}{2} Z_2\, |\vec{H}_t|^2\, \hat{k}_t$$

$$\vec{\mathcal{P}}_{avg_2} = \frac{1}{2}\,\frac{377}{3}\,|144{,}98|^2\left[\sin\theta_2\,\hat{\imath}_x + \cos\theta_2\,\hat{\imath}_z\right]$$

$$= 1{,}3207 \cdot 10^6 \left(\frac{W}{m^2}\right)\left[0.1666\,\hat{\imath}_x + 0{,}986\,\hat{\imath}_z\right]$$

(στ)

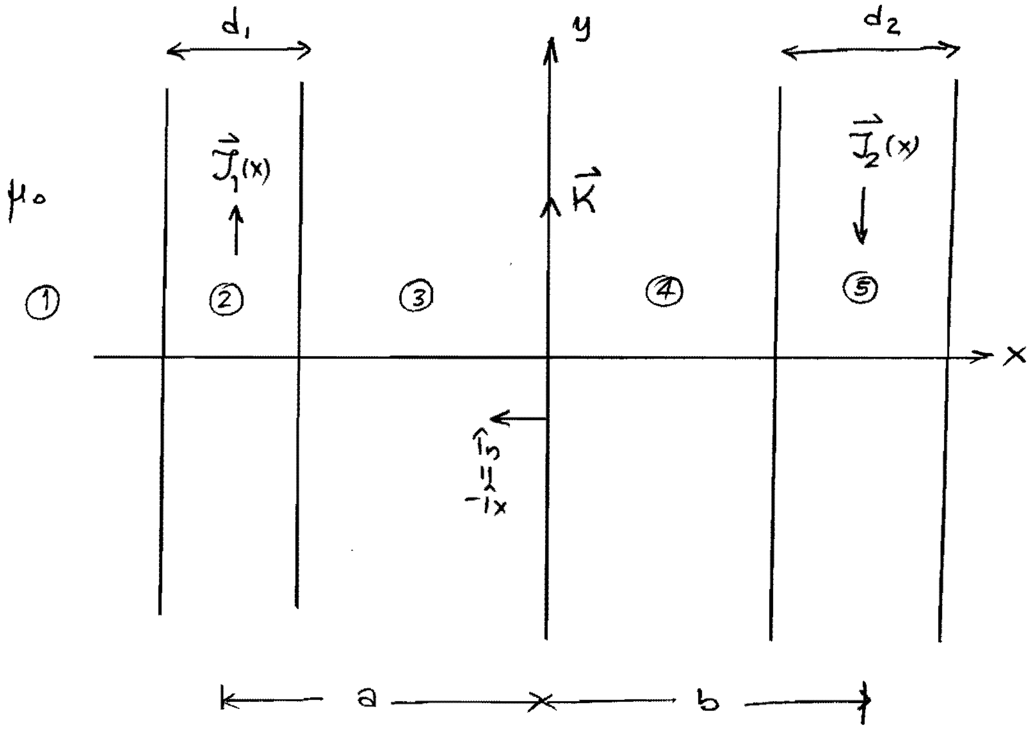
$$P_r = |r_{TM}|^2 = 0.2023$$

$$P_t = \frac{Z_1}{Z_2}\,\frac{\cos\theta_2}{\cos\theta_1}\,|t_{TM}|^2 = \frac{3\cos\theta_2}{1\cos\theta_1}\,|t_{TM}|^2 =$$

$$= 0.7977$$

Άρα το 20.23% ανακλάται και το 79,77% διαδίδεται.

ΑΣΚΗΣΗ 7:



$$\vec{J}_1(x) = J_{01}\cos(\pi(x+a)/d_1)\,\hat{i}_y$$

$$\vec{J}_2(x) = -J_{02}\cos(\pi(x-b)/d_2)\,\hat{i}_y \qquad \vec{K} = K_0\,\hat{i}_y$$

$$(\alpha)\quad \vec{H}_1 = \frac{\vec{K}\times\hat{i}_n}{2}(+1) + \int_{-a-\frac{d_1}{2}}^{-a+\frac{d_1}{2}} \frac{J_1(x)dx\;\hat{i}_y\times\hat{i}_n}{2}(+1) + \int_{b-\frac{d_2}{2}}^{b+\frac{d_2}{2}} \frac{J_2(x)dx\,(-\hat{i}_y)\times\hat{i}_n}{2}(+1)$$

$$\hat{i}_y\times\hat{i}_n = \hat{i}_y\times(-\hat{i}_x) = -\hat{i}_y\times\hat{i}_x = \hat{i}_z$$

Άρα $$\vec{H}_1 = \hat{i}_z\left\{\frac{K_0}{2} + \frac{1}{2}\int_{-a-\frac{d_1}{2}}^{-a+d_1/2} J_{01}\cos\left(\pi\frac{x+a}{d_1}\right)dx - \frac{1}{2}\int_{b-\frac{d_2}{2}}^{b+\frac{d_2}{2}} J_{02}\cos\left(\pi\frac{x-b}{d_2}\right)dx\right\}$$

$$\int_{-a-d_1/2}^{-a+d_1/2} J_{01}\cos\left(\pi\frac{x+a}{d_1}\right)dx = J_{01}\left(\frac{\sin\left(\pi\frac{x+a}{d_1}\right)}{\pi/d_1}\right)\Bigg|_{-a-d_1/2}^{-a+d_1/2} = J_{01}\frac{2d_1}{\pi}$$

$$\int_{+b-d_2/2}^{b+d_2/2} J_{02}\cos\left(\pi\frac{x-b}{d_2}\right)dx = J_{02}\frac{2d_2}{\pi}$$

$$\vec{H}_1 = \hat{i}_z\left\{\frac{K_0}{2} + J_{01}\frac{\cancel{2}d_1}{\pi}\frac{1}{\cancel{2}} - J_{02}\frac{\cancel{2}d_2}{\pi}\frac{1}{\cancel{2}}\right\}$$

Όμοια για την ⑥ περιοχή:

$$\vec{H}_6 = \frac{\vec{K}\times\hat{i}_n}{2}(-1) + \frac{1}{2}\int (\quad)(-1)\,dx + \frac{1}{2}\int(\quad)(-1)\,dx =$$

$$= -\vec{H}_1 = -\hat{i}_z\left\{\frac{K_o}{2} + J_{o1}\frac{\cancel{2}d_1}{\pi}\frac{1}{\cancel{2}} - J_{o2}\frac{\cancel{2}d_2}{\pi}\frac{1}{\cancel{2}}\right\}$$

$$H_1 = 0 = H_6 \rightarrow \underline{\frac{K_o}{2} + \frac{J_{o1}d_1}{\pi} - \frac{J_{o2}d_2}{\pi} = 0}$$

(β) $-a-\frac{d_1}{2} < x < b+\frac{d_2}{2}$: περιοχές ②,③,④,⑤

$$\vec{H}_2 = \frac{K_o}{2}\hat{i}_z - \hat{i}_z\int_{-a-d_1/2}^{x}\frac{1}{2}J_{o1}\cos\left(\pi\frac{(x+a)}{d_1}\right)dx + \hat{i}_z\int_{x}^{-a+d_1/2}\frac{1}{2}J_{o1}\cos\left(\pi\frac{(x+a)}{d_1}\right)dx$$

$$-\hat{i}_z\int_{b-\frac{d_2}{2}}^{b+\frac{d_2}{2}}\frac{1}{2}J_{o2}\cos\left(\pi\frac{x-b}{d_2}\right)dx =$$

$$= \hat{i}_z\left\{\frac{K_o}{2} - \frac{1}{2}J_{o1}\frac{d_1}{\pi}\left\{\sin\left(\pi\frac{x+a}{d_1}\right)+1\right\} + J_{o1}\frac{1}{2}\frac{d_1}{\pi}\left\{1-\sin\left(\pi\frac{x+a}{d_1}\right)\right\}\right.$$

$$\left. -\frac{1}{2}J_{o2}\frac{2d_2}{\pi}\right\}$$

$$= \hat{i}_z\left\{\frac{K_o}{2} - \frac{J_{o1}d_1}{\pi}\sin\left(\pi\frac{x+a}{d_1}\right) - J_{o2}\frac{d_2}{\pi}\right\}$$

$$\vec{H}_3 = \hat{i}_z\left\{\frac{K_o}{2} - J_{o1}\frac{d_1}{\pi} - J_{o2}\frac{d_2}{\pi}\right\} = \hat{i}_z\left(-\frac{2J_{o1}d_1}{\pi}\right)$$

$$\vec{H}_4 = \hat{i}_z\left\{-\frac{K_o}{2} - J_{o1}\frac{d_1}{\pi} - J_{o2}\frac{d_2}{\pi}\right\} = \hat{i}_z\left(-\frac{2J_{o2}d_2}{\pi}\right)$$

$$\vec{H}_5 = \hat{i}_z\left\{-\frac{K_o}{2} - J_{o1}\frac{1}{2}\frac{2d_1}{\pi} + \frac{1}{2}\int_{b-\frac{d_2}{2}}^{x}J_{o2}\cos\left(\pi\frac{x-b}{d_2}\right)dx - \frac{1}{2}\int_{x}^{b+d_2/2}J_{o2}\cos\left(\pi\frac{x-b}{d_2}\right)dx\right\}$$

$$= \hat{i}_z\left\{-\frac{K_o}{2} - J_{o1}\frac{d_1}{\pi} + J_{o2}\frac{d_2}{2\pi}\left(\sin\left(\pi\frac{x-b}{d_2}\right)+1\right)\right.$$

$$\left. - J_{o2}\frac{d_2}{2\pi}\left(1-\sin\left(\pi\frac{x-b}{d_2}\right)\right)\right\}$$

$$= \hat{i}_z\left\{-\frac{K_o}{2} - J_{o1}\frac{d_1}{\pi} + J_{o2}\frac{d_2}{\pi}\sin\left(\pi\frac{x-b}{d_2}\right)\right\}$$