49RHUW: HYDRA June 15-16, a.k.a. Dimitrios & Friends 2023

Sensitivity of Nonlinear Dynamic Response
&
Relative Importance of Input Variables

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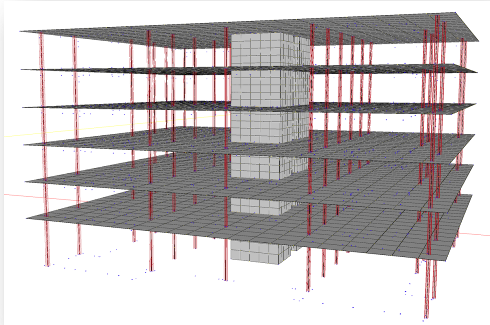
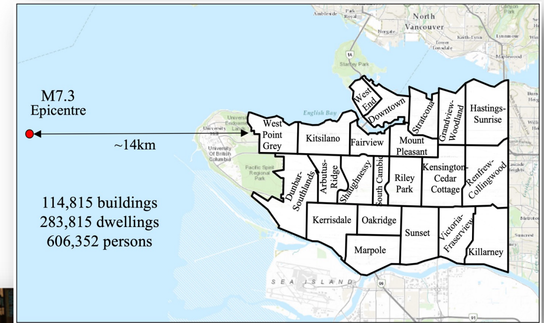
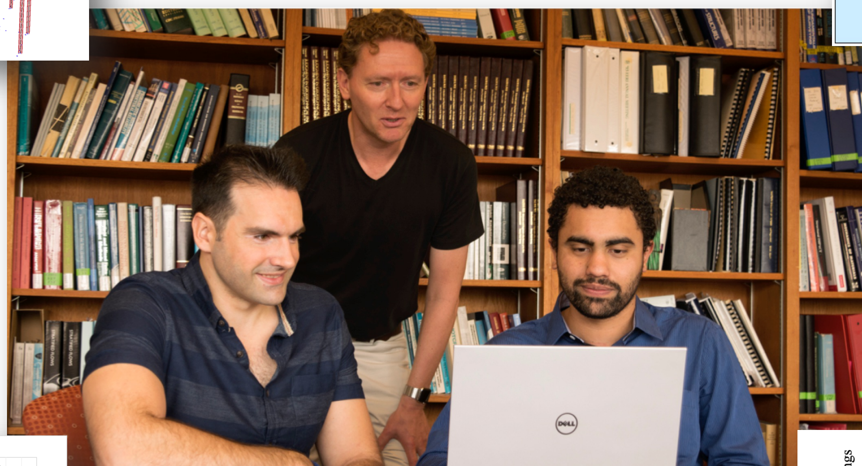
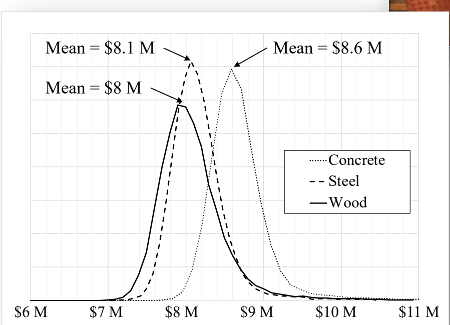
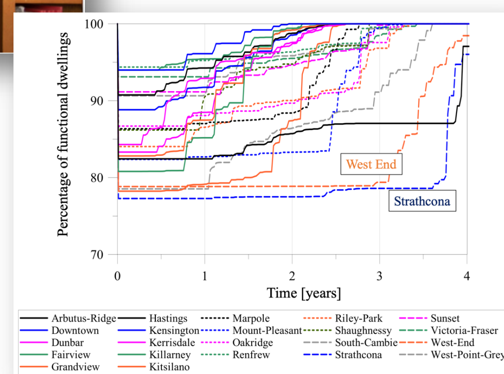
Back to the Future





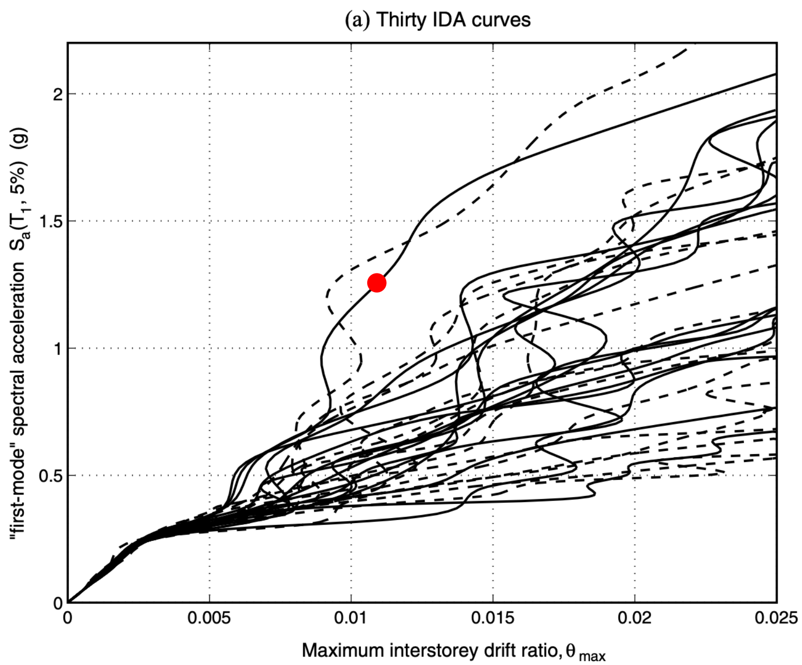






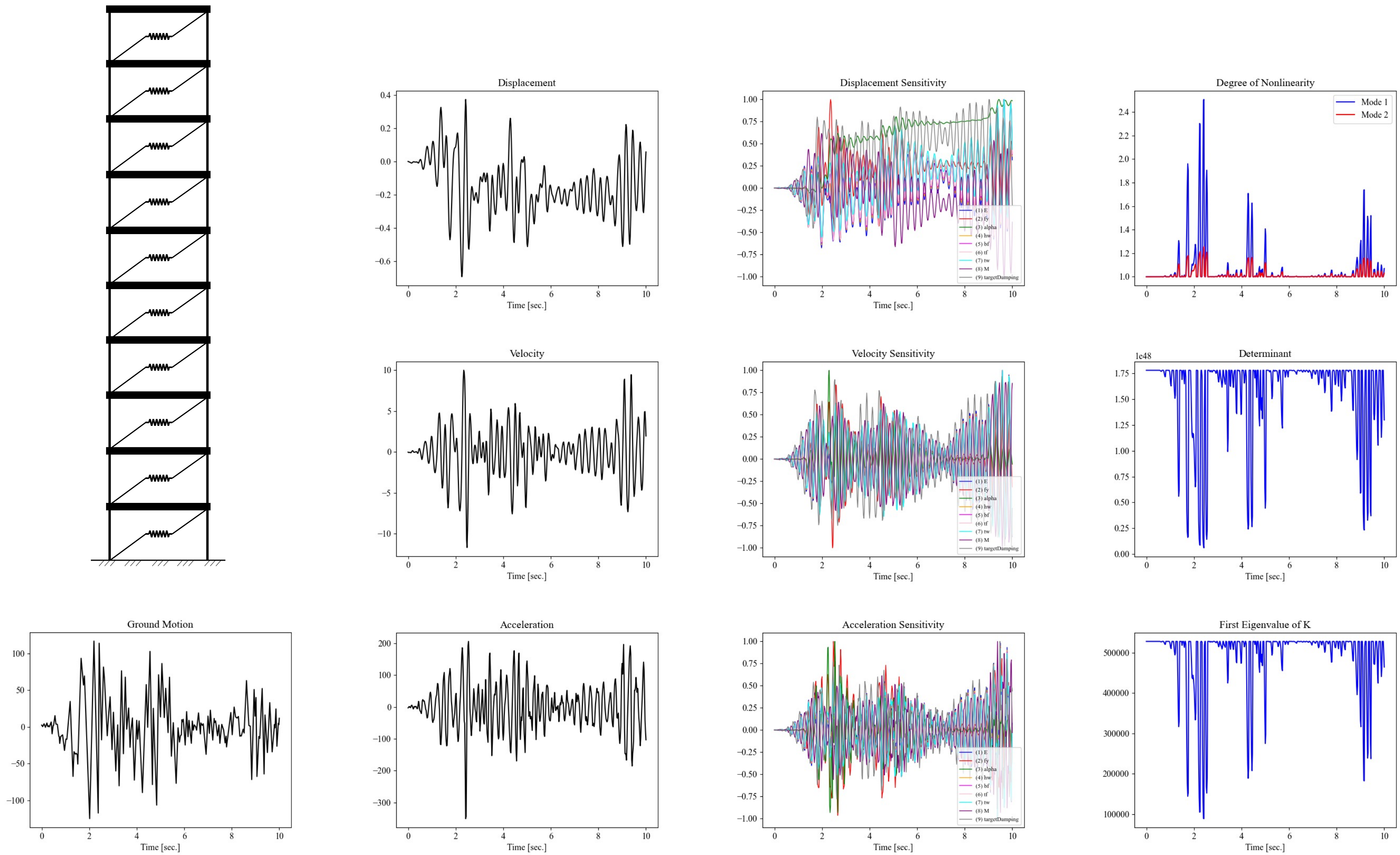
Nonlinear Dynamics

D. VAMVATSIKOS AND C. A. CORNELL



Each Analysis

What to Monitor



Today

How to **calculate** response sensitivities (from COMPDYN 2023 paper)

How to **read** response sensitivities

How to **rank** input variables using response sensitivities (from ICASP14 paper)

Calculate

Direct Differentiation Method

Exact & efficient 👍

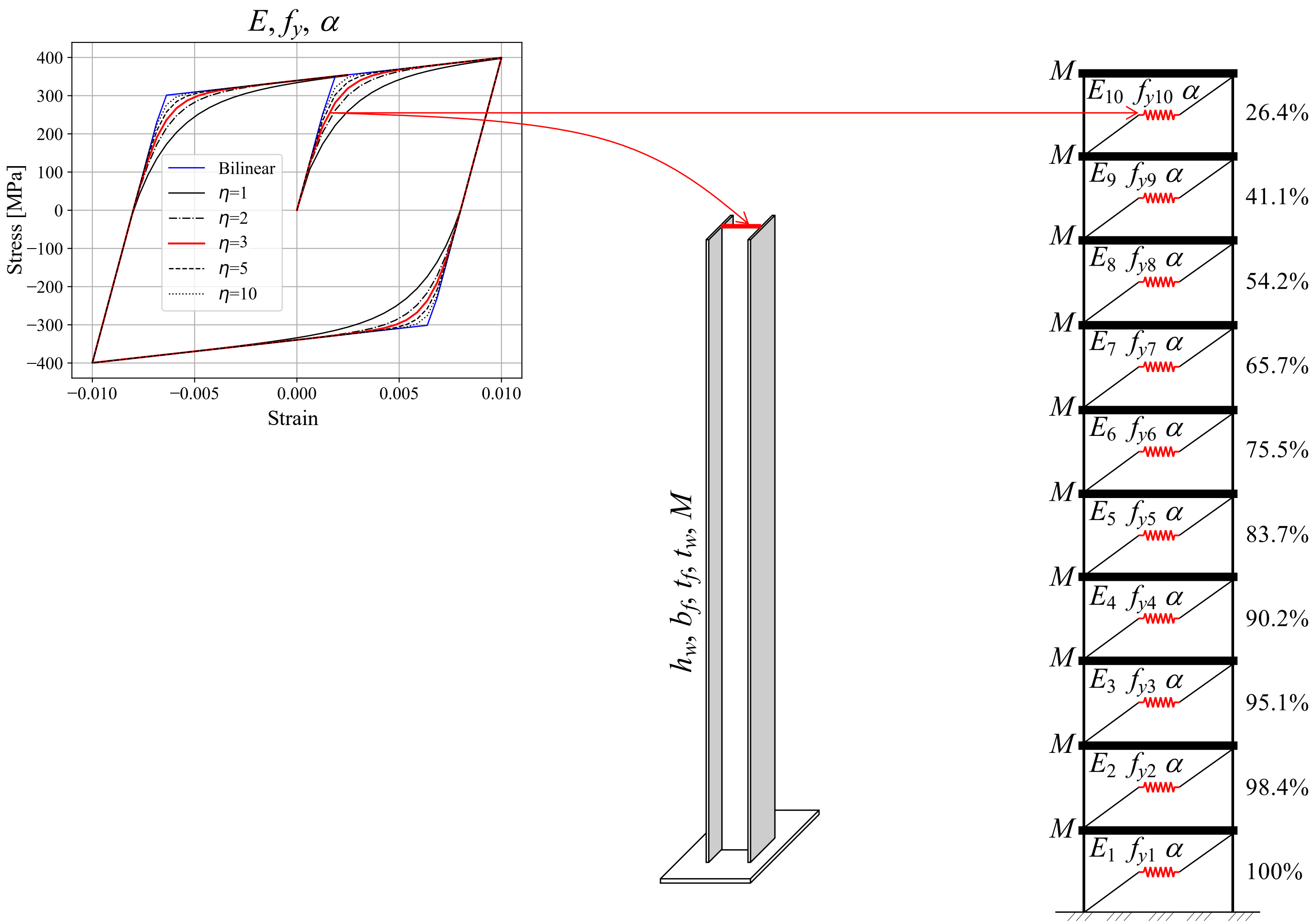
No need to re-run the analysis 👍

One-time effort to differentiate & implement 👎 🤓

Early work: University of Iowa & University of California at Berkeley

Here: New damping terms

Parameters



+ damping, ζ

Equation of Motion

Depend on $\mathbf{u}_{n+1}$

$$\frac{\partial}{\partial x}\bigg| \left(\mathbf{M}\,\ddot{\mathbf{u}}_{n+1} + \mathbf{C}_{n+1}\dot{\mathbf{u}}_{n+1} + \tilde{\mathbf{F}}_{n+1} = \mathbf{F}_{n+1} \right)$$

Internal resisting forces

Material, geometry, load

Everything evaluated at time t_{n+1}

Governing Equation for $\partial \mathbf{u}_{n+1}/\partial x$

$$\frac{\partial \mathbf{M}}{\partial x}\ddot{\mathbf{u}}_{n+1} + \mathbf{M}\frac{\partial \ddot{\mathbf{u}}_{n+1}}{\partial x} + \frac{\partial \mathbf{C}_{n+1}}{\partial x}\dot{\mathbf{u}}_{n+1} + \mathbf{C}_{n+1}\frac{\partial \dot{\mathbf{u}}_{n+1}}{\partial x} + \frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial x} = \frac{\partial \mathbf{F}_{n+1}}{\partial x}$$

New in this presentation

$$\frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial x} = \frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial \mathbf{u}_{n+1}} \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial x} + \left.\frac{\partial \tilde{\mathbf{F}}}{\partial x}\right|_{\mathbf{u}_{n+1}\ \text{fixed}}$$

$$\frac{\partial \mathbf{C}_{n+1}}{\partial x} = \frac{\partial \mathbf{C}_{n+1}}{\partial \mathbf{K}_{n+1}} \cdot \frac{\partial \mathbf{K}_{n+1}}{\partial \mathbf{u}_{n+1}} \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial x} + \left.\frac{\partial \mathbf{C}}{\partial x}\right|_{\mathbf{u}_{n+1}\ \text{fixed}}$$

Fourth-order tensor

Third-order tensor

1. Implicit dependence via $\mathbf{u}$
2. Explicit dependence via algorithm for $\tilde{\mathbf{F}}$
3. Conditional derivatives in “Phase 1”
4. Not necessarily fixed strain
5. Unconditional derivatives in “Phase 2” (Zhang & Der Kiureghian 1993)

Temporal Discretization

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\tilde{\beta}\Delta t^2} \cdot \mathbf{u}_{n+1} - \frac{1}{\tilde{\beta}\Delta t^2} \cdot \mathbf{u}_n - \frac{1}{\tilde{\beta}\Delta t} \cdot \dot{\mathbf{u}}_n + \left(1 - \frac{1}{2\tilde{\beta}}\right) \cdot \ddot{\mathbf{u}}_n$$

$$\mathbf{M}\,\ddot{\mathbf{u}}_{n+1} + \mathbf{C}_{n+1}\dot{\mathbf{u}}_{n+1} + \tilde{\mathbf{F}}_{n+1} = \mathbf{F}_{n+1}$$

$$\dot{\mathbf{u}}_{n+1} = \frac{\tilde{\gamma}}{\tilde{\beta}\Delta t} \cdot \mathbf{u}_{n+1} - \frac{\tilde{\gamma}}{\tilde{\beta}\Delta t} \cdot \mathbf{u}_n - \left(1 - \frac{\tilde{\gamma}}{\tilde{\beta}}\right) \cdot \dot{\mathbf{u}}_n + \Delta t \cdot \left(1 - \frac{\tilde{\gamma}}{2\tilde{\beta}}\right) \cdot \ddot{\mathbf{u}}_n$$

Sort Everything

$$\left[\text{Coefficient matrix}\right] \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial x} = \text{Right-hand side}$$

↑

Usually identical to the coefficient matrix for $\mathbf{u}_{n+1}$,
i.e., the effective dynamic stiffness

Coefficient Matrix

$$\frac{1}{\tilde{\beta}\Delta t^2}\cdot \mathbf{M}\frac{\partial \mathbf{u}_{n+1}}{\partial x} + \frac{\tilde{\gamma}}{\tilde{\beta}\Delta t}\cdot \mathbf{C}_{n+1}\frac{\partial \mathbf{u}_{n+1}}{\partial x} + \left(\frac{\partial \mathbf{C}_{n+1}}{\partial \mathbf{K}_{n+1}}\frac{\partial \mathbf{K}_{n+1}}{\partial \mathbf{u}_{n+1}}\frac{\partial \mathbf{u}_{n+1}}{\partial x}\dot{\mathbf{u}}_{n+1}\right) + \frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial \mathbf{u}_{n+1}}\frac{\partial \mathbf{u}_{n+1}}{\partial x} = \text{Right-hand side}$$

Ordinary effective dynamic stiffness

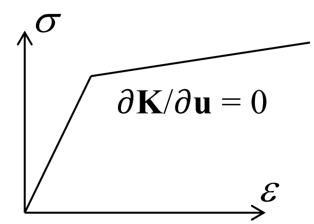
$$\left[\frac{1}{\tilde{\beta}\Delta t^2}\cdot M_{ij} + \frac{\tilde{\gamma}}{\tilde{\beta}\Delta t}\cdot C_{ij} + \underbrace{\left(\frac{\partial C_{im}}{\partial K_{op}}\frac{\partial K_{op}}{\partial u_j}v_m\right)} + \frac{\partial \tilde{F}_i}{\partial u_j}\right]\frac{\partial u_j}{\partial x} = \text{Right-hand side}$$

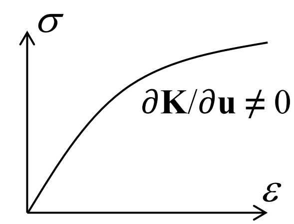
New in this presentation

The Amendment $\frac{\partial C_{im}}{\partial K_{op}} \frac{\partial K_{op}}{\partial u_j} v_m$

- Parameter-independent

- The third-order tensor is non-zero only for certain material models:





- The index of the velocity contracts with the second index of the forth-order tensor

- If $\partial \mathbf{C}/\partial \mathbf{K}=\mathbf{1}$ then the velocity contracts with the middle index of the third-order tensor

- Index notation matches computer implementation: `einsum('imop,opj,m->ij', dCdK, dKdu, v)`

Derivatives

What We Need

$$\frac{\partial \mathbf{M}}{\partial x}\ddot{\mathbf{u}}_{n+1} + \mathbf{M}\frac{\partial \ddot{\mathbf{u}}_{n+1}}{\partial x} + \frac{\partial \mathbf{C}_{n+1}}{\partial x}\dot{\mathbf{u}}_{n+1} + \mathbf{C}_{n+1}\frac{\partial \dot{\mathbf{u}}_{n+1}}{\partial x} + \frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial x} = \frac{\partial \mathbf{F}_{n+1}}{\partial x}$$

$$\frac{\partial \mathbf{C}_{n+1}}{\partial x} = \frac{\partial \mathbf{C}_{n+1}}{\partial \mathbf{K}_{n+1}} \cdot \frac{\partial \mathbf{K}_{n+1}}{\partial \mathbf{u}_{n+1}} \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial x} + \left.\frac{\partial \mathbf{C}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}}$$

$$\frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial x} = \frac{\partial \tilde{\mathbf{F}}_{n+1}}{\partial \mathbf{u}_{n+1}} \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial x} + \left.\frac{\partial \tilde{\mathbf{F}}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}}$$

Nonlinear frame elements

Fiber-discretized cross-sections

$$\frac{\partial k_{n+1}}{\partial \varepsilon_{n+1}} \qquad \left.\frac{\partial k_{n+1}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}} \qquad k_{n+1} \equiv \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} \qquad \left.\frac{\partial \sigma_{n+1}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}}$$

Eigenvalue derivatives

Algorithmically consistent tangent

Modal Damping

$$C_{ij} = \frac{2\zeta\omega}{m} M_{ik}\phi_k\phi_l M_{lj}$$

$$\frac{\partial C_{ik}}{\partial K_{op}} = \frac{\partial C_{ik}}{\partial m}\frac{\partial m}{\partial \phi_q}\frac{\partial \phi_q}{\partial K_{op}} + \frac{\partial C_{ik}}{\partial \omega}\frac{\partial \omega}{\partial K_{op}} + \frac{\partial C_{ik}}{\partial \phi_q}\frac{\partial \phi_q}{\partial K_{op}}$$

$$\frac{\partial C_{ik}}{\partial m} = -\frac{2\zeta\omega}{m^2} \cdot M_{il}\phi_l\phi_n M_{nk}$$

$$\frac{\partial m}{\partial \phi_q} = M_{qt}\phi_t + \phi_s M_{sq}$$

$$\frac{\partial C_{ik}}{\partial \omega} = \frac{2\zeta}{m} \cdot M_{il}\phi_l\phi_n M_{nk}$$

$$\frac{\partial C_{ik}}{\partial \phi_q} = \frac{2\zeta\omega}{m} \cdot \left(M_{iq}\phi_n M_{nk} + M_{il}\phi_l M_{qk}\right)$$

$$\left.\frac{\partial C_{ij}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}} = \frac{\partial \omega}{\partial x} \cdot \frac{2\zeta}{m} \cdot M_{ik}\phi_k\phi_l M_{lj} - \frac{\partial m}{\partial x} \cdot \frac{2\zeta\omega}{m^2} \cdot M_{ik}\phi_k\phi_l M_{lj} + \frac{2\zeta\omega}{m} \cdot \frac{\partial M_{ik}}{\partial x}\phi_k\phi_l M_{lj} + \frac{2\zeta\omega}{m} \cdot M_{ik}\phi_k\phi_l\frac{\partial M_{lj}}{\partial x} + \frac{2\zeta\omega}{m} \cdot M_{ik}\frac{\partial \phi_k}{\partial x}\phi_l M_{lj} + \frac{2\zeta\omega}{m} \cdot M_{ik}\phi_k\frac{\partial \phi_l}{\partial x}M_{lj}$$

$$\frac{\partial m}{\partial x} = \frac{\partial \phi_i}{\partial x}M_{ij}\phi_j + \phi_i\frac{\partial M_{ij}}{\partial x}\phi_j + \phi_i M_{ij}\frac{\partial \phi_j}{\partial x}$$

Rayleigh Damping

$$C_{ij} = \left(\frac{2\zeta \cdot \breve{\omega} \cdot \hat{\omega}}{\breve{\omega} + \hat{\omega}}\right) \cdot M_{ij} + \left(\frac{2\zeta}{\breve{\omega} + \hat{\omega}}\right) \cdot K_{ij}$$

$$\frac{\partial C_{ij}}{\partial K_{op}} = \frac{\partial \breve{\omega}}{\partial K_{op}} \cdot \frac{2\zeta \cdot \hat{\omega}}{\breve{\omega} + \hat{\omega}} \cdot M_{ij} + \frac{\partial \hat{\omega}}{\partial K_{op}} \cdot \frac{2\zeta \cdot \breve{\omega}}{\breve{\omega} + \hat{\omega}} \cdot M_{ij} - \frac{2\zeta \cdot \breve{\omega} \cdot \hat{\omega}}{(\breve{\omega} + \hat{\omega})^2}\left(\frac{\partial \breve{\omega}}{\partial K_{op}} + \frac{\partial \hat{\omega}}{\partial K_{op}}\right)M_{ij} - \frac{2\zeta}{(\breve{\omega} + \hat{\omega})^2}\left(\frac{\partial \breve{\omega}}{\partial K_{op}} + \frac{\partial \hat{\omega}}{\partial K_{op}}\right)K_{ij} + \frac{2\zeta}{\breve{\omega} + \hat{\omega}} \cdot \frac{\partial K_{ij}}{\partial K_{op}}$$

$$\left.\frac{\partial C_{ij}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}} = \frac{\partial \zeta}{\partial x} \cdot \frac{2 \cdot \breve{\omega} \cdot \hat{\omega}}{\breve{\omega} + \hat{\omega}} \cdot M_{ij} + \frac{\partial \zeta}{\partial x} \cdot \frac{2}{\breve{\omega} + \hat{\omega}} \cdot K_{ij} + \frac{\partial \breve{\omega}}{\partial x} \cdot \frac{2\zeta \cdot \hat{\omega}}{\breve{\omega} + \hat{\omega}} \cdot M_{ij} + \frac{\partial \hat{\omega}}{\partial x} \cdot \frac{2\zeta \cdot \breve{\omega}}{\breve{\omega} + \hat{\omega}} \cdot M_{ij} - \frac{2\zeta \cdot \breve{\omega} \cdot \hat{\omega}}{(\breve{\omega} + \hat{\omega})^2}\left(\frac{\partial \breve{\omega}}{\partial x} + \frac{\partial \hat{\omega}}{\partial x}\right)M_{ij} - \frac{2\zeta}{(\breve{\omega} + \hat{\omega})^2}\left(\frac{\partial \breve{\omega}}{\partial x} + \frac{\partial \hat{\omega}}{\partial x}\right)K_{ij} + \frac{2\zeta \cdot \breve{\omega} \cdot \hat{\omega}}{\breve{\omega} + \hat{\omega}} \cdot \frac{\partial M_{ij}}{\partial x} + \frac{2\zeta}{\breve{\omega} + \hat{\omega}} \cdot \left.\frac{\partial K_{ij}}{\partial x}\right|_{\mathbf{u}_{n+1}\text{ fixed}}$$

Eigenvalue Derivatives

$$\left(K_{ij} - \gamma M_{ij}\right)\phi_j = 0_i$$

$$\frac{\partial \gamma}{\partial x} = \frac{\phi_i\frac{\partial K_{ij}}{\partial x}\phi_j - \gamma\phi_i\frac{\partial M_{ij}}{\partial x}\phi_j}{\phi_i M_{ij}\phi_j} \qquad \frac{\partial \gamma}{\partial K_{kl}} = \frac{\phi_k\phi_l}{\phi_i M_{ij}\phi_j}$$

$$\frac{\partial \phi_j}{\partial x} = \left(K_{ij} - \gamma M_{ij}\right)^{+}\left(-\frac{\partial K_{ij}}{\partial x} + \frac{\partial \gamma}{\partial x}M_{ij} + \gamma\frac{\partial M_{ij}}{\partial x}\right)\phi_j \qquad \frac{\partial \phi_j}{\partial K_{kl}} = -\left(K_{ij} - \gamma M_{ij}\right)^{+}\left(\frac{\partial K_{ij}}{\partial K_{kl}} - \frac{\partial \gamma}{\partial K_{kl}}M_{ij}\right)\phi_j$$

Element-level

$$\mathbf{K}^{\mathrm{G}} = \left(\mathbf{T}^{\mathrm{BG}}\right)^{\mathrm{T}}\mathbf{K}^{\mathrm{B}}\mathbf{T}^{\mathrm{BG}}$$

$$\frac{\partial K^{\mathrm{G}}_{il}}{\partial u^{\mathrm{G}}_m} = T^{\mathrm{BG}}_{ji} \cdot \frac{\partial K^{\mathrm{B}}_{jk}}{\partial u^{\mathrm{G}}_m} \cdot T^{\mathrm{BG}}_{kl} = T^{\mathrm{BG}}_{ji} \cdot \left(\frac{\partial K^{\mathrm{B}}_{jk}}{\partial u^{\mathrm{B}}_p} \cdot \frac{\partial u^{\mathrm{B}}_p}{\partial u^{\mathrm{G}}_m}\right) \cdot T^{\mathrm{BG}}_{kl} = T^{\mathrm{BG}}_{ji} \cdot T^{\mathrm{BG}}_{pm} \cdot T^{\mathrm{BG}}_{kl} \cdot \frac{\partial K^{\mathrm{B}}_{jk}}{\partial u^{\mathrm{B}}_p}$$

Section-level

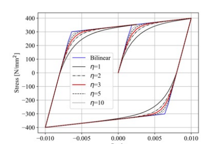
$$\left.\frac{\partial \tilde{F}^{\mathrm{S}}_i}{\partial x}\right|_{u^{\mathrm{S}}_{n+1}\text{ fixed}} = \frac{\partial T^{\mathrm{MatS}}_i}{\partial x} \cdot A \cdot \sigma + T^{\mathrm{MatS}}_i \cdot \frac{\partial A}{\partial x} \cdot \sigma + T^{\mathrm{MatS}}_i \cdot A \cdot \frac{\partial \sigma}{\partial x} = \frac{\partial T^{\mathrm{MatS}}_i}{\partial x} \cdot A \cdot \sigma + T^{\mathrm{MatS}}_i \cdot \frac{\partial A}{\partial x} \cdot \sigma + T^{\mathrm{MatS}}_i \cdot A \cdot \left(\frac{\partial \sigma}{\partial \varepsilon_{n+1}} \cdot \left.\frac{\partial \varepsilon_{n+1}}{\partial x}\right|_{u^{\mathrm{S}}_{n+1}\text{ fixed}} + \left.\frac{\partial \sigma}{\partial x}\right|_{\varepsilon_{n+1}\text{ fixed}}\right)$$

$$\left.\frac{\partial K^{\mathrm{S}}_{ij}}{\partial x}\right|_{u^{\mathrm{S}}_{n+1}\text{ fixed}} = \frac{\partial T^{\mathrm{MatS}}_i}{\partial x} \cdot T^{\mathrm{MatS}}_j \cdot A \cdot k + T^{\mathrm{MatS}}_i \cdot \frac{\partial T^{\mathrm{Mat}}_j}{\partial x} \cdot A \cdot k + T^{\mathrm{MatS}}_i \cdot T^{\mathrm{MatS}}_j \cdot \frac{\partial A}{\partial x} \cdot k + T^{\mathrm{MatS}}_i \cdot T^{\mathrm{MatS}}_j \cdot A \cdot \left(\frac{\partial k}{\partial \varepsilon_{n+1}} \cdot \left.\frac{\partial \varepsilon_{n+1}}{\partial x}\right|_{u^{\mathrm{S}}_{n+1}\text{ fixed}} + \left.\frac{\partial k}{\partial x}\right|_{\varepsilon_{n+1}\text{ fixed}}\right)$$

$$\frac{\partial K^{\mathrm{S}}_{ij}}{\partial u^{\mathrm{S}}_k} = T^{\mathrm{MatS}}_i \cdot T^{\mathrm{MatS}}_j \cdot A \cdot \frac{\partial k}{\partial u_k} = T^{\mathrm{MatS}}_i \cdot T^{\mathrm{MatS}}_j \cdot A \cdot \left(\frac{\partial k}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial u_k}\right) = T^{\mathrm{MatS}}_i \cdot T^{\mathrm{MatS}}_j \cdot T^{\mathrm{MatS}}_k \cdot A \cdot \frac{\partial k}{\partial \varepsilon}$$

Material-level: Bouc-Wen

$$\sigma_{n+1} = \alpha \cdot E \cdot \varepsilon_{n+1} + (1 - \alpha) \cdot f_y \cdot z_{n+1}$$



$$k_{n+1} = \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = \alpha \cdot E + (1 - \alpha) \cdot f_y \cdot \frac{\partial z_{n+1}}{\partial \varepsilon_{n+1}}$$

$$\dot{z} = \left(\dot{\varepsilon} - \gamma \cdot \dot{\varepsilon} \cdot |z|^{\eta} - \beta \cdot |\dot{\varepsilon}| \cdot z^{\eta}\right) \cdot \frac{1}{\varepsilon_y} = \left(1 - (\gamma + \beta \cdot \mathrm{sgn}(\dot{\varepsilon} \cdot z) \cdot |z|^{\eta}\right)\frac{\dot{\varepsilon}}{\varepsilon_y}$$

$$z_{n+1} = z_n + \left(1 - (\gamma + \beta \cdot \mathrm{sgn}(\dot{\varepsilon}_{n+1} \cdot z_n)) \cdot |z_{n+1}|^{\eta}\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y}$$

Bouc-Wen Derivatives

$$k_{n+1} = \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} =$$

$$\frac{\partial z_{n+1}}{\partial \varepsilon_{n+1}} =$$

$$\left.\frac{\partial k_{n+1}}{\partial x}\right|_{\varepsilon_{n+1}\text{ fixed}} =$$

$$\left.\frac{\partial^2 z_{n+1}}{\partial \varepsilon_{n+1}\partial x}\right|_{\varepsilon_{n+1}\text{ fixed}} =$$

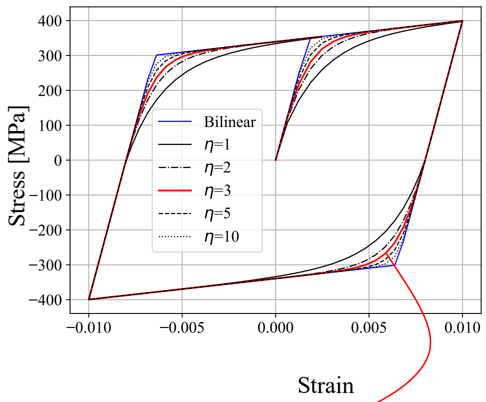
$$\left.\frac{\partial \sigma_{n+1}}{\partial x}\right|_{\varepsilon_{n+1}\text{ fixed}} =$$

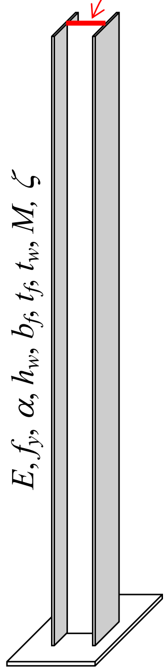
$$\left.\frac{\partial z_{n+1}}{\partial x}\right|_{\varepsilon_{n+1}\text{ fixed}} =$$

$$\frac{\partial k_{n+1}}{\partial \varepsilon_{n+1}} =$$

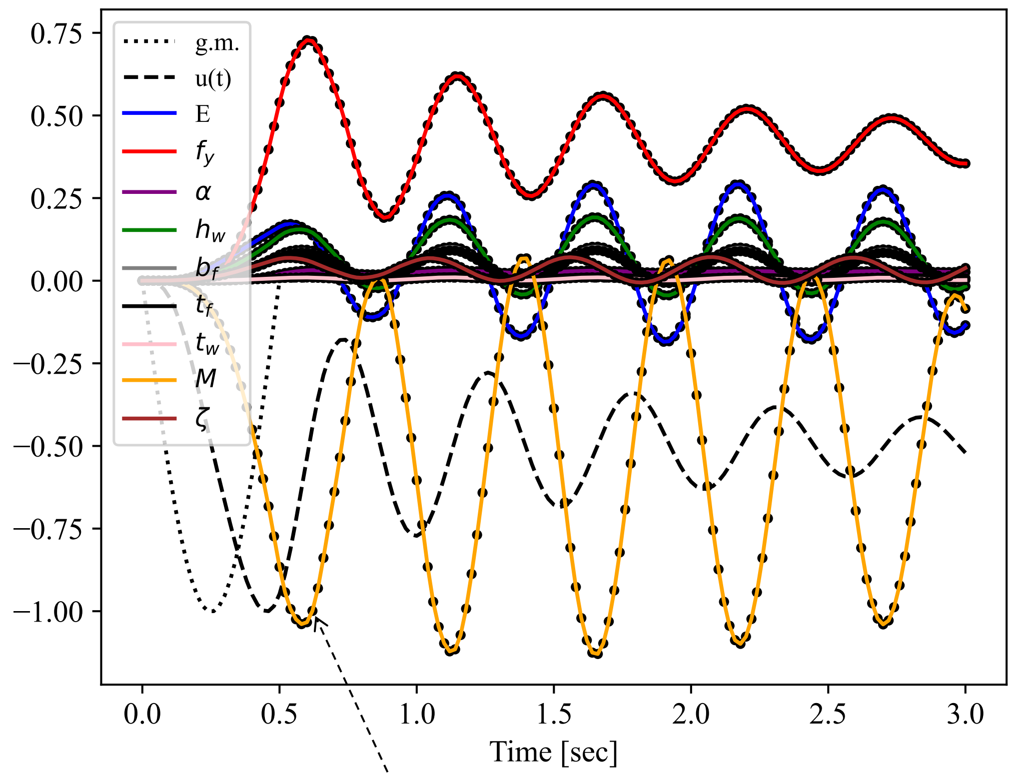
$$\frac{\partial^2 z_{n+1}}{\partial \varepsilon^2_{n+1}} =$$

Implementation & Verification





Python code freely available at **terje.civil.ubc.ca**

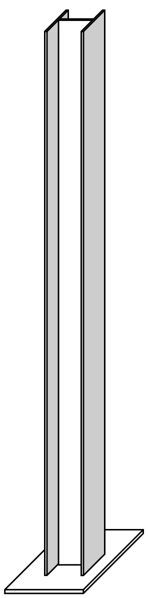


- Extra runs for finite difference calculations

Read

Growth of Peak(s)

1. The score at time 2.42sec is 0.6160 with gradient contribution 0.2441
2. The score at time 1.38sec is 0.5576 with gradient contribution 0.2332
3. The score at time 2.08sec is 0.2841 with gradient contribution 0.1129
4. The score at time 1.60sec is 0.2388 with gradient contribution 0.0841
5. The score at time 0.80sec is 0.1318 with gradient contribution 0.0408
6. The score at time 1.04sec is 0.1225 with gradient contribution 0.0018
7. The score at time 0.52sec is 0.0128 with gradient contribution 0.0007
8. The score at time 0.24sec is 0.0006 with gradient contribution 0.0026
9. The score at time 0.36sec is -0.0025 with gradient contribution 0.0007
10. The score at time 2.68sec is -0.0762 with gradient contribution 0.0744
11. The score at time 2.86sec is -0.0964 with gradient contribution 0.0909

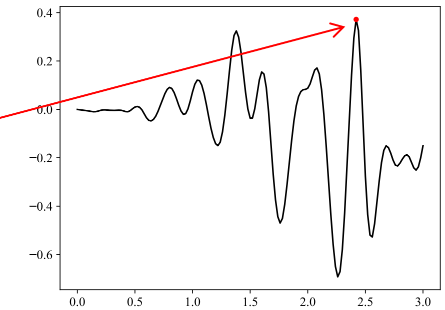


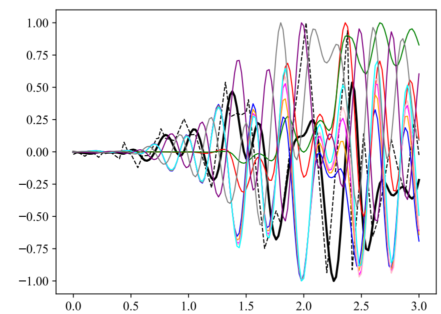
Ranking for peak at t=2.42 (u=0.37):

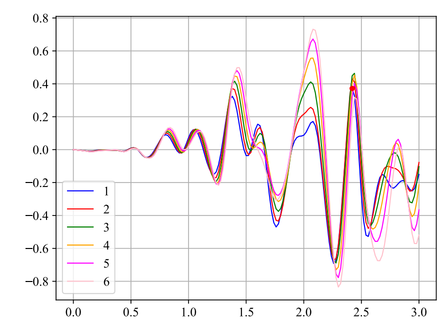
1.	h_w	0.798
2.	E	0.441
3.	M	0.239
4.	b_f	0.235
5.	f_y	0.201
6.	t_f	0.121
7.	α	0.016
8.	ζ	0.009
9.	t_w	0.005

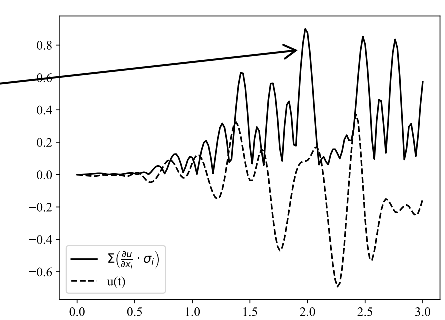
Look at $\sum \frac{\partial u}{\partial x_i} \cdot \sigma_i$

El Centro ground motion









Rank

Basic Importance Measure

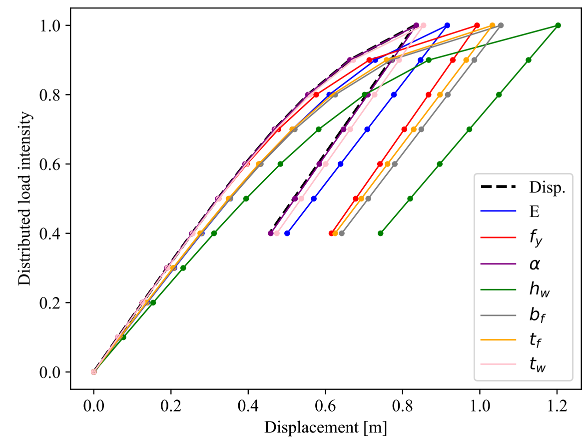
Response sensitivities, $\partial u/\partial x_i$, cannot be compared

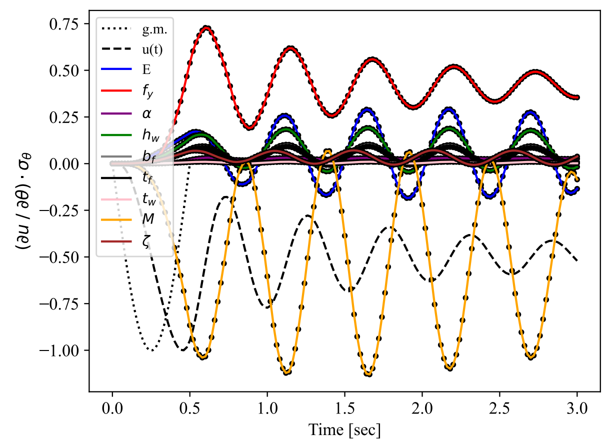
Basic importance measure:

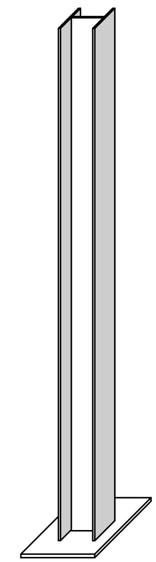
$$\frac{\partial u}{\partial x_i} \cdot \sigma_i$$

Standard deviation $\sigma_i = \delta_i \cdot \mu_i$

Coefficient of variation   Mean



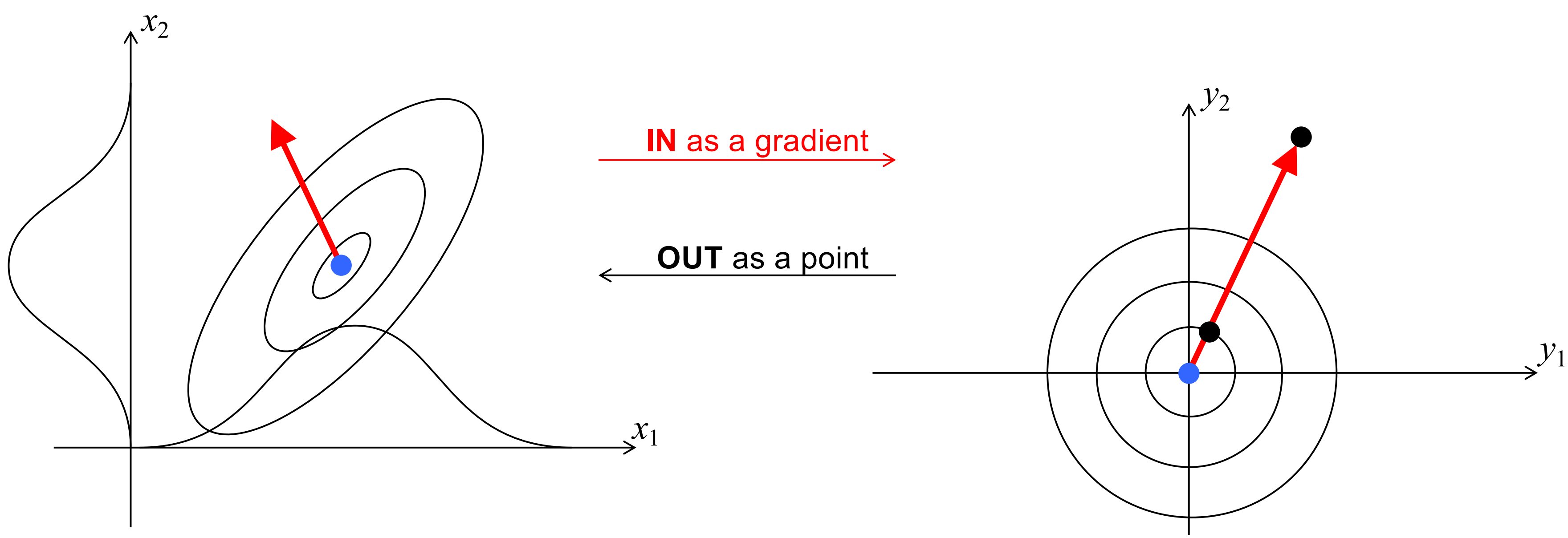




Include Correlation?

Original parameter space:

Space of standard uncorrelated variables:



Second-moment Transformation

$$\mathbf{x} = \mathbf{M} + \mathbf{DLy}$$

Vector in standard space: $$\frac{\partial u}{\partial \mathbf{y}} = \frac{\partial u}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial u}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{DL}$$

End point in original space: $$\mathbf{x} = \mathbf{M} + \mathbf{DL} \underbrace{\frac{\partial u}{\partial \mathbf{y}}}_{!}$$

Input Variable Perturbations

$$\Delta \mathbf{x}_{\mathrm{V}} = \boxed{\mathbf{DL}\frac{\partial u}{\partial \mathbf{y}}}$$

$\Delta \mathbf{x}_{\mathrm{V}}$ is the most likely change in $\mathbf{x}$ to cause a response-change equal to the response variance for linear models

$$\Delta \mathbf{x}_{\sigma} = \frac{1}{||\frac{\partial u}{\partial \mathbf{x}}\mathbf{DL}||} \cdot \overbrace{\mathbf{DL}\left(\frac{\partial u}{\partial \mathbf{x}}\mathbf{DL}\right)}$$

$\Delta \mathbf{x}_{\sigma}$ is the most likely change in $\mathbf{x}$ to cause a response-change equal to the response standard deviation for linear models

Importance Measures

$$\boldsymbol{\tau}_{\sigma,\mathrm{i}} = \frac{\Delta \mathbf{x}_{\sigma,i}}{\sigma_i}$$

$\boldsymbol{\tau}_{\sigma}$ cannot be directly interpreted as contributions from the **variance** of each variable to the total variance of the response

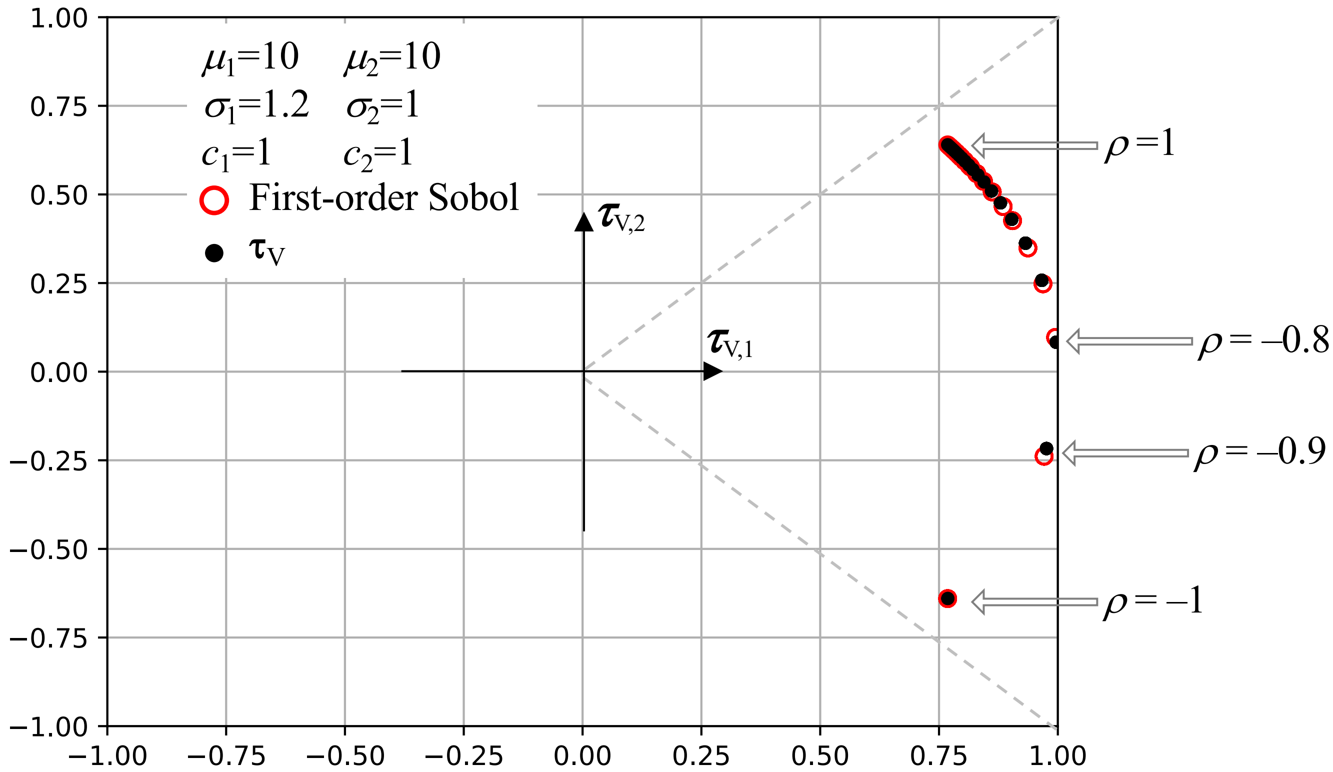
---

$$\boldsymbol{\tau}_{\mathrm{V}} = \Delta \mathbf{x}_{\mathrm{V}} \odot \frac{\partial u}{\partial \mathbf{x}}$$

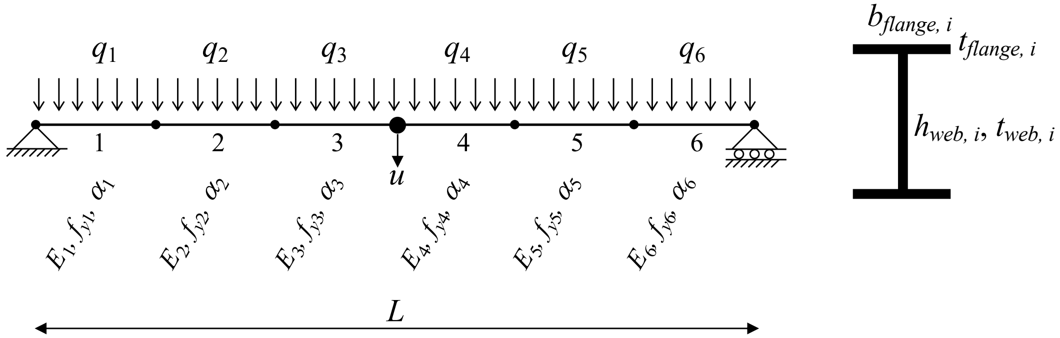
$\boldsymbol{\tau}_{\mathrm{V}}$ ranks the variables according their contribution to the total variance of the response, while simultaneously accounting for correlation, because:

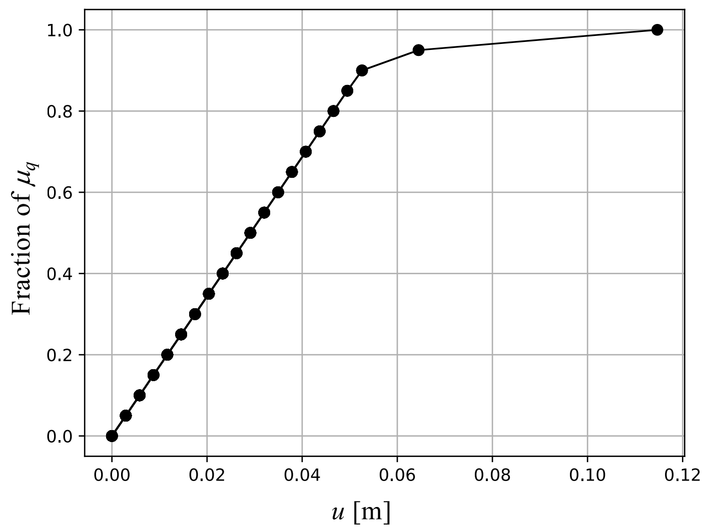
$$\sum_{i=1}^{N} \boldsymbol{\tau}_{\mathrm{V},i} = \frac{\partial u}{\partial \mathbf{x}}^{\mathrm{T}} \boldsymbol{\Sigma} \frac{\partial u}{\partial \mathbf{x}} = \sigma_u^2$$

τ_V vs. Sobol for $u = c_1 \cdot x_1 + c_2 \cdot x_2$

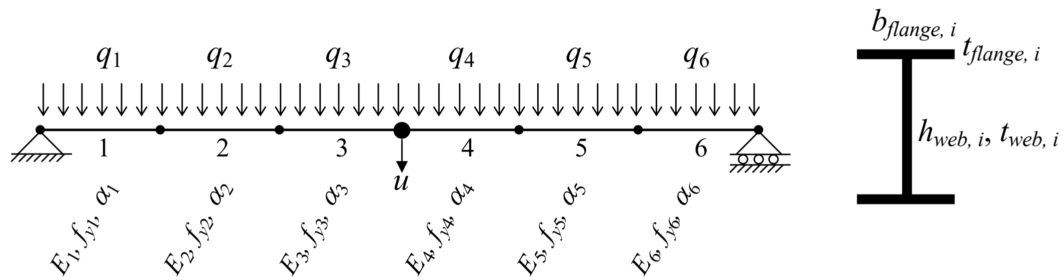


Example


q_1 q_2 q_3 q_4 q_5 q_6
1 2 3 4 5 6
u
E_1, f_{y1}, α_1
E_2, f_{y2}, α_2
E_3, f_{y3}, α_3
E_4, f_{y4}, α_4
E_5, f_{y5}, α_5
E_6, f_{y6}, α_6
L
$b_{flange,\,i}$
$t_{flange,\,i}$
$h_{web,\,i}, t_{web,\,i}$

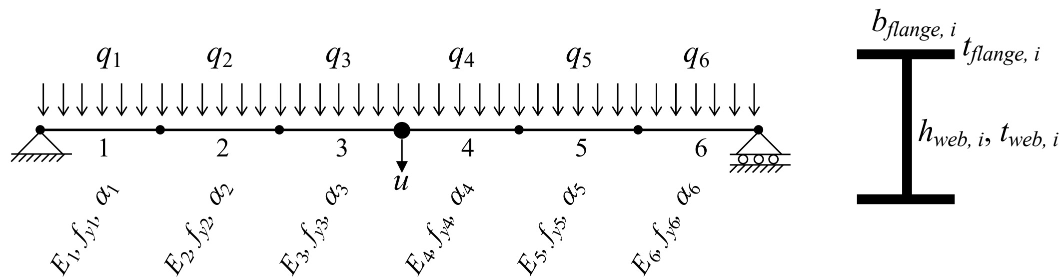

Fraction of μ_q
1.0
0.8
0.6
0.4
0.2
0.0
0.00 0.02 0.04 0.06 0.08 0.10 0.12
u [m]

Linear Analysis



	Case 1		Case 2		Case 3		Case 4	
1	q, El. 4	0.44	E, El. 4	0.53	q, El. 4	0.55	h_w, El. 4	0.56
2	q, El. 3	0.44	E, El. 3	0.53	q, El. 3	0.55	h_w, El. 3	0.56
3	h_w, El. 4	0.38	q, El. 4	0.27	q, El. 5	0.38	h_w, El. 5	0.25
4	h_w, El. 3	0.38	q, El. 3	0.27	q, El. 2	0.38	h_w, El. 2	0.25
5	E, El. 4	0.31	E, El. 5	0.24	h_w, El. 4	0.14	q, El. 4	0.24
6	E, El. 3	0.31	E, El. 2	0.24	h_w, El. 3	0.14	q, El. 3	0.24
7	q, El. 5	0.22	h_w, El. 4	0.24	q, El. 6	0.13	E, El. 4	0.17
8	q, El. 2	0.22	h_w, El. 3	0.24	q, El. 1	0.13	E, El. 3	0.17
9	h_w, El. 5	0.08	q, El. 5	0.14	E, El. 4	0.12	q, El. 5	0.12
10	h_w, El. 2	0.08	q, El. 2	0.14	E, El. 3	0.12	q, El. 2	0.12

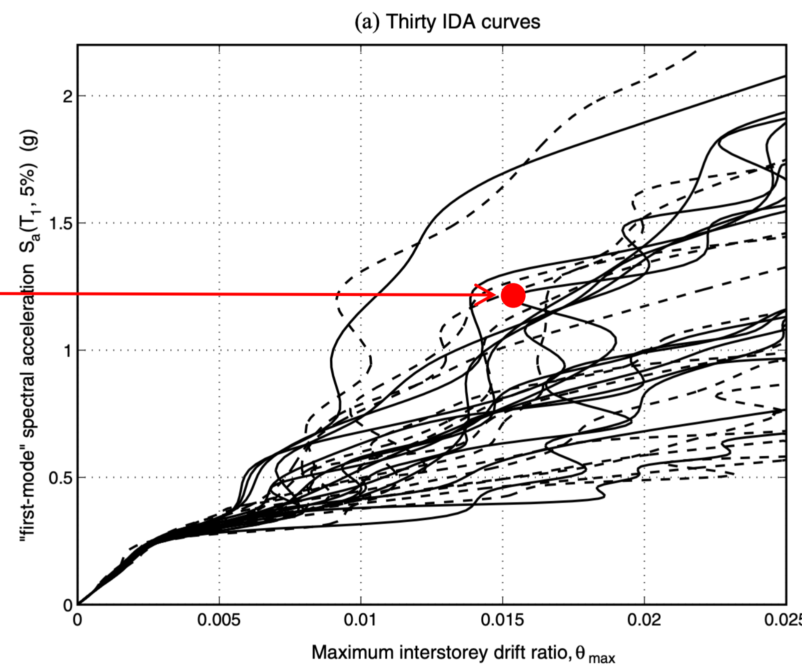
Nonlinear Analysis



	Case 1		Case 2		Case 3		Case 4	
1	f_y, El. 3	0.68	f_y, El. 3	0.70	f_y, El. 3	0.56	f_y, El. 3	0.67
2	f_y, El. 4	0.68	f_y, El. 4	0.70	f_y, El. 4	0.56	f_y, El. 4	0.67
3	q, El. 3	0.14	q, El. 3	0.08	q, El. 3	0.36	h_w, El. 3	0.14
4	q, El. 4	0.14	q, El. 4	0.08	q, El. 4	0.36	h_w, El. 4	0.14
5	h_w, El. 3	0.08	h_w, El. 3	0.04	q, El. 2	0.22	q, El. 3	0.14
6	h_w, El. 4	0.08	h_w, El. 4	0.04	q, El. 5	0.22	q, El. 4	0.14
7	q, El. 2	0.05	q, El. 2	0.03	q, El. 1	0.07	b_f, El. 3	0.07
8	q, El. 5	0.05	q, El. 5	0.03	q, El. 6	0.07	b_f, El. 4	0.07
9	b_f, El. 3	0.04	b_f, El. 3	0.02	h_w, El. 3	0.06	t_f, El. 3	0.06
10	b_f, El. 4	0.04	b_f, El. 4	0.02	h_w, El. 4	0.06	t_f, El. 4	0.06

Learn from Sensitivities!

D. VAMVATSIKOS AND C. A. CORNELL



Thank You for Your Attention!