Lecture 33: Legendre Polynomials and Spherical Harmonics

1 Laplacian in Spherical Coordinates

The Laplacian in spherical coordinates is

$$\nabla^2 u(r, \theta, \phi) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}. \quad (1)$$

This can be separated by writing

$$u = R(r) \Theta(\theta) \Phi(\phi),$$

so that,

$$r^2 \frac{\nabla^2 u}{u} = \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2}. \quad (2)$$

1.1 $\phi$ dependence

First, we separate out the $\phi$ dependence with a separation constant $m$,

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi, \quad (3)$$

which is a Sturm-Liouville equation with $0 < \phi < 2\pi$ boundary conditions $\Phi(0) = \Phi(2\pi)$ and $\Phi'(0) = \Phi'(2\pi)$. The eigenfunctions and eigenvalues are

$$\Phi_m(\phi) = e^{im\phi}, \quad m = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \quad (4)$$

Now the $\phi$ dependence in the Laplacian has been replaced by $m$,

$$r^2 \frac{\nabla^2 u}{u} = \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta}. \quad (5)$$

1.2 $\theta$ dependence

Next we separate out the $\theta$ dependence with a separation constant $l(l + 1)$,

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2 \Theta}{\sin^2 \theta} = -l(l + 1) \Theta. \quad (6)$$

This can be simplified by changing variables $x = \cos \theta$. In fact, we get a Sturm-Liouville equation,

$$[(1 - x^2)y']' + \left[ l(l + 1) - \frac{m^2}{1 - x^2} \right] y = 0, \quad (7)$$
with boundary conditions $y(1) = 1$ and $y(-1) = \pm 1$.

Now all the angular $\theta$ and $\phi$ dependence in the Laplacian has been replaced by $l$,

$$r^2 \frac{\nabla^2 u}{u} = \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - l(l + 1).$$  \hspace{1cm} (8)

## 2 Legendre Equation

When $m = 0$, the equation for $\theta$ dependence, Eq. (7), becomes the Legendre equation,

$$[(1 - x^2)y']' + l(l + 1)y = 0,$$ \hspace{1cm} (9)

This equation has two series solutions (even and odd) with recursion relation

$$a_{n+2} = \frac{n(n + 1) - l(l + 1)}{(n + 1)(n + 2)} a_n.$$ \hspace{1cm} (10)

This series solution converges for $-1 < x < 1$, and diverges at $x = \pm 1$.

The eigenfunctions of the Legendre equation that are well behaved at $x = \pm 1$ exist for special values of $l$ that truncate the series solution to a finite polynomial. From the recursion relation we see that integer values of $l \geq 0$ will truncate the expansion with $a_{l+2} = 0$. Thus the eigenvalues are $l = 0, 1, 2, \ldots$, and the eigenfunctions are the Legendre polynomials, $P_l(x)$. By convention, the Legendre polynomials are normalized so that $P_l(1) = 1$. Since Legendre polynomials are even or odd, this implies $P_l(-1) = (-1)^l$.

### 2.1 Rodrigues’ formula

Although the recursion relation Eq. (10) can be used to find the Legendre polynomials, a more useful expression is Rodrigues’ formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$ \hspace{1cm} (11)

### 2.2 Normalization of Legendre Polynomials

From Sturm-Liouville theory, we know that Legendre polynomials are orthogonal. Using Rodrigues’ formula and the convention $P_l(1) = 1$ we can determine that the orthogonality and normalization condition

$$\int_{-1}^{1} P_l(x) P_k(x) dx = \frac{2}{2l + 1} \delta_{lk}.$$ \hspace{1cm} (12)
2.3 Generating function for Legendre Polynomials

Another representation of Legendre polynomials is to use a generating function \( G(x, h) \) such that \( P_n(x) \) are the coefficients for a power series expansion of \( G(x, h) \) in the dummy variable \( h \). This generating function is

\[
G(x, h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{n=0}^{\infty} P_n(x) h^n.
\]

This generating function representation is very useful in physical applications when \( x \) is identified as the cosine of the angle between two points, and \( h < 1 \) is the ratio of their distances. Then

\[
\frac{1}{|r - r'|} = \sum_{n=0}^{\infty} P_n(\cos \theta) \frac{r_<}{r_>},
\]

This expression can be used for writing the electrostatic potential of a point charge in polar coordinates.

3 Associated Legendre Equations

For \( m \neq 0 \), the associated Legendre equation, Eq. (7) has eigenvalues \( l = |m|, |m| + 1, |m| + 2, \ldots \). The associated Legendre polynomials are related to the Legendre polynomials by

\[
P_l^m(x) = (1 - x^2)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} P_l(x).
\]

Their orthogonality and normalization is

\[
\int_{-1}^{1} P_l^m(x) P_k^m(x) dx = \frac{2}{2l + 1} \frac{(l + m)!}{(l - m)!} \delta_{lk},
\]

which reduces to the normalization for Legendre polynomials when \( m = 0 \).

4 Spherical harmonics

The spherical harmonics for \( m \geq 0 \) are

\[
Y_l^m(\theta, \phi) = (-1)^m \left[ \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \right] P_l^m(\cos \theta) e^{im\phi},
\]

with \( l = 0, 1, 2, \ldots \) and integer \( m \) in the range \(-l \leq m \leq l\). The coefficients have been chosen to make the spherical harmonics orthonormal,

\[
\int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi [Y_l^m(\theta, \phi)]^* Y_{l'}^{m'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}.
\]