Lecture 33: Legendre Polynomials and Spherical Harmonics

1 Laplacian in Spherical Coordinates

The Laplacian in spherical coordinates is

$$\nabla^2 u(r,\theta,\phi) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}. \tag{1}$$

This can be separated by writing $u = R(r)\Theta(\theta)\Phi(\phi)$, so that,

$$r^2\frac{\nabla^2 u}{u} = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}. \tag{2}$$

1.1 ϕ dependence

First, we separate out the ϕ dependence with a separation constant m,

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi, \tag{3}$$

which is a Sturm-Liouville equation with $0 < \phi < 2\pi$ boundary conditions $\Phi(0) = \Phi(2\pi)$ and $\Phi'(0) = \Phi'(2\pi)$. The eigenfunctions and eigenvalues are

$$\Phi_m(\phi) = e^{im\phi}, \qquad m = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \tag{4}$$

Now the ϕ dependence in the Laplacian has been replaced by m,

$$r^2\frac{\nabla^2 u}{u} = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}. \tag{5}$$

1.2 θ dependence

Next we separate out the θ dependence with a separation constant $l(l+1)$,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2\Theta}{\sin^2\theta} = -l(l+1)\Theta. \tag{6}$$

This can be simplified by changing variables $x = \cos\theta$. In fact, we get a Sturm-Liouville equation,

$$[(1-x^2)y']' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0, \tag{7}$$

with boundary conditions $y(1) = 1$ and $y(-1) = \pm 1$.

Now all the angular θ and ϕ dependence in the Laplacian has been replaced by l,

$$r^2 \frac{\nabla^2 u}{u} = \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - l(l+1). \tag{8}$$

2 Legendre Equation

When $m = 0$, the equation for θ dependence, Eq. (7), becomes the Legendre equation,

$$[(1-x^2)y']' + l(l+1)y = 0, \tag{9}$$

This equation has two series solutions (even and odd) with recursion relation

$$a_{n+2} = \frac{[n(n+1) - l(l+1)]}{(n+1)(n+2)} a_n. \tag{10}$$

This series solution converges for $-1 < x < 1$, and diverges at $x = \pm 1$.

The eigenfunctions of the Legendre equation that are well behaved at $x = \pm 1$ exist for special values of l that truncate the series solution to a finite polynomial. From the recursion relation we see that integer values of $l \geq 0$ will truncate the expansion with $a_{l+2} = 0$. Thus the eigenvalues are $l = 0, 1, 2, \ldots$, and the eigenfunctions are the Legendre polynomials, $P_l(x)$. By convention, the Legendre polynomials are normalized so that $P_l(1) = 1$. Since Legendre polynomials are even or odd, this implies $P_l(-1) = (-1)^l$.

2.1 Rodrigues' formula

Although the recursion relation Eq. (10) can be used to find the Legendre polynomials, a more useful expression is Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(x^2-1)^l. \tag{11}$$

2.2 Normalization of Legendre Polynomials

From Sturm-Liouville theory, we know that Legendre polynomials are orthogonal. Using Rodrigues' formula and the convention $P_l(1) = 1$ we can determine that the orthogonality and normalization condition

$$\int_{-1}^{1} P_l(x)P_k(x)dx = \frac{2}{2l+1}\delta_{lk}. \tag{12}$$

2.3 Generating function for Legendre Polynomials

Another representation of Legendre polynomials is to use a generating function $G(x,h)$ such that $P_n(x)$ are the coefficents for a power series expansion of $G(x,h)$ in the dummy variable h. This generating function is

$$G(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{n=0}^{\infty} P_n(x)h^n. \tag{13}$$

This generating function representation is very useful in physical applications when x is identified as the cosine of the angle between two points, and $h < 1$ is the ratio of their distances. Then

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{r_<}{r_>}. \tag{14}$$

This expression can be used for writing the electrostatic potential of a point charge in polar coordinates.

3 Associated Legendre Equtation

For $m \neq 0$, the associated Legendre equation, Eq. (7) has eigenvalues $l = |m|, |m|+1, |m|+2, \ldots$. The associated Legendre polynomials are related to the Legendre polynomials by

$$P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x). \tag{15}$$

Their orthogonality and normalization is

$$\int_{-1}^{1} P_l^m(x)P_k^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{lk}, \tag{16}$$

which reduces to the normalization for Legendre polynomials when $m = 0$.

4 Spherical harmonics

The spherical harmonics for $m \geq 0$ are

$$Y_l^m(\theta,\phi) = (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right] P_l^m(\cos\theta)e^{im\phi}, \tag{17}$$

with $l = 0, 1, 2, \ldots$ and integer m in the range $-l \leq m \leq l$. The coefficents have been chosen to make the spherical harmonics orthonormal,

$$\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi [Y_l^m(\theta,\phi)]^* Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{18}$$