CAD-FEA integration using Coons interpolation

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Abstract

This paper reviews recent advances and presents new findings towards an attempt to integrate FEA with solid modeling. The objective is to handle any sufficiently smooth mechanical component by control lines of its boundary only, for both purposes: geometry description and structural/multi-physics analysis. First, it is shown that bivariate Coons interpolation is capable of developing two-dimensional large finite macroelements without any internal nodes, as well as patches of large three-dimensional boundary elements. Second, it is reminded that trivariate Coons interpolation is capable of generating three-dimensional finite element meshes within boxlike regions, for which a new smoothening procedure is here proposed for the first time. Finally, it is shown that trivariate Coons interpolation is also capable of developing large three-dimensional finite macroelements with the nodal points over the boundary only and –in many cases– along the twelve edges of the solid region (considered as a curvilinear paralleloidal), which can properly adapted to each mechanical component. Aspects of scientific visualization and differences from NURBS representation are also discussed.

1. Introduction

Integration between different communities seems to be a strategic aim nowadays. As an example, geometric modeling (CAD) and computer-aided analysis (CAE) are usually individually powerful, but they do not always work well together. In addition to that, integration between geometric design and scientific visualization or between CAE and visualization is not a trivial procedure. Within the last years, some solutions have been proposed by using trivariate NURBS as a *unifying* representation. Also, the topic of different resolution requirements between the geometrical and visualization model in order to achieve large savings in storage and execution time, have been discussed [1,2].

A vehicle to achieve CAD/CAE integration is to apply a common basis function for geometric modeling and representation of the multi-physics field (temperature, displacement, etc.). Casale [3] and coworkers [4,5] proposed trimmed surface patches as boundary elements. At the same time, Kanarachos and Deriziotis [6] were influenced by the ideas of Gordon and Hall [57] and developed a Coons-based boundary-type method that used cubic B-splines interpolation and was applied to solve 2-D boundary-value potential problems. In that primary work, it was found that the proposed method leads to better results than FEM and BEM for both static and dynamic analysis. Previously, a Coons-patch method including boundary derivatives had been proposed to differential equations with predominant lower-order derivatives [7]. Similar ideas were also applied to plate bending problems [8-10]. Also, the author of this paper has later contributed to the promotion and further extension of these ideas [20-33].

Moreover, Kagan and Fisher [11] developed a B-spline based finite element scheme. Renken and Subbarayan [12] used NURBS to represent the shape of droplets by integrating surface energy coefficients over appropriate surfaces. Henshaw [13] introduced an alternative method for addressing the problem of CAD and FEM integration. The method deals with the problem of generating structured meshes over CAD models defined by a large set of trimmed NURBS patches. It is addressed by a newly developed technique for fast projection of points onto a patched CAD model.

Upon finishing the writing of this paper, it was discovered that even the latest issues of CAD/CAE journals include several contributions towards the abovementioned integration. Clark and Anderson [14,15] propose a penalty boundary method for performing finite element analysis using a regular overlapping mesh that does not have to coincide with the geometric boundaries. Previously, Charlesworth et al. [16] have proposed a 'domain decomposition' technique aiming to relax or remove the restriction on the mesh to conform globally to the domain geometry. In addition, Natekar et al. [17] proposed an NURBS-based analysis methodology that is procedurally analogous to the Constructive Solid Geometry (CSG) integrating design and analysis, and thereby enabling efficient optimal design. This method was applied to two-dimensional problems only.

According to Natekar et al. [17], bivariate NURBS representation is applied to derive shape functions $N_I(\xi,\eta)$ that are based on the set of *I*-th control points defining the system geometry. The same shape functions are also used to approximate the dispacement field within the domain. It is remarkable that at any point within the domain the sum of these shape functions equals to the unity but the value of $N_I(\xi,\eta)$ at the *I*-th control point is not unity to that node. Therefore, "*even if the control point were to be coincident with the location of the boundary condition, direct application of the boundary condition is not possible since the specified field value will be distributed to control points influencing the point under consideration*" [17].

On the contrary, since 1982 the CAD/CAE group at NTUA has adopted a different philosophy using the Coons-interpolation [6,43] instead of NURBS. Of course, there is no doubt that NURBS is superior to Coons' interpolation formula [18] as the first is capable of representing sculptured surfaces and offers *local* control while the second is a rather *global* interpolation, probably excepting the case of using bubble functions [19]. Nevertheless, Coons-interpolation can easily deal with nodes along the boundary of the domain and the obtained global shape functions equal to unity at them [6,27-29], a fact that makes an essential difference with similar NURBS-based techniques such as that in the above paragraph mentioned within quotation marks [17]. As a result, the proposed Coons-patch methodology is directly applicable using all standard finite element procedures in both static and dynamics regimes, without being necessary to use penalty methods [14,15] and related Lagrange-multipliers techniques [17]. Clearly, in the proposed methodology no internal nodes are required, even in the sense of meshless [58] and mesh-free [59] techniques.

In more details, it has been recently shown that the *bivariate* Coons interpolation offers the possibility to develop 2-D large finite elements, called "Coons-patch macroelements" that are characterized by degrees of freedom appearing along the boundary of the component only, and not within its area. Concretely, bivariate formula has been used to develop finite macroelements for plane problems [20-29], both static [20,21,24,27-29] and dynamic [20,22-27]. A careful study of this literature reveals that these macroelements are very accurate and in many cases of higher quality than conventional finite elements and boundary elements. Also, bivariate interpolation has been used to develop large boundary elements occupying extended curvilinear patches in order to solve three-dimensional elastic structures [30,31], acoustic enclosures [32] and sound radiation problems [33].

It is remarkable that prior to that, bivariate Coons interpolation had been used to generate two-dimensional finite element meshes for plane-stress, plane-strain and plate bending problems [34-37]. The interested reader may also consult Reference [38], where details are provided for mesh generation in both finite elements (FEM) and boundary element (BEM) applications.

Moreover, *trivariate* Coons interpolation is capable of describing any boxlike solid component on the basis of its boundary [18, p.41]. Thirty years ago, this formula was used by Cook [39] to develop a generator for three-dimensional computational mesh, useful for further FEM analysis.

Apart from the above review aspects, there are two novel features of this paper. Briefly, it is shown that the abovementioned trivariate Coons-interpolation offers the two following alternative possibilities in the integration between CAE and FEM:

- It achieves to generate *smooth* meshes within boxlike structures.
- It achieves to avoid the mesh generation and, instead of that, it can work in conjunction with only the twelve edges of the boxlike structure. As it was earlier mentioned, the latter has been achieved for the case of large three-dimensional boundary elements [30-33], but here, it will be extended also to three-dimensional large finite *macroelements*. As it was previously mentioned, this methodology appears two major advantages. First, it operates directly on the geometric modeling representation and, second, it preserves the FEM formulation in both static and dynamic analyses. Especially, in dynamic analysis the BEM has several shortcomings [40-42] and therefore an alternative solution is needed.

Following to the above introduction, a summary of the overall applicability of Coons' interpolation is schematically shown in **Figure 1**.

Concerning boxlike structures, we can assume that they consist of a curvilinear paralleloidal made of six surfaces, eight corners and twelve edges. Therefore, the above trivariate Coons approach offers to designers a powerful tool in order to reduce data preparation and shape optimization costs, as they will have to handle with only a limited number of parameters, that is the control points along the twelve sides of the mechanical component. Obviously, for complicated components it will be necessary to use control points above the whole surface or to divide the model into a small number of large-size boxlike regions.

Since the term "*Coons-patch*" clearly describes curvilinear surfaces, in the sequence, the term "*superbrick*" will be introduced to describe boxlike volumes. Again, superbrick generally refers to a curvilinear paralleloidal made of six surfaces, eight corners and twelve edges. Following to above, within this article the term "superbrick" will refer to either CAD (geometry representation) or CAE (trivariate attribute model) activities.

This paper is structured as follows. In section 2, bivariate and trivariate Coons-formulas are presented in terms of lofting projections. In section 3, three alternative formulations of trivariate Coons interpolation are presented for the first time. In section 4, a smoothening procedure is proposed for both two- and three-dimensional regions and its applications field is discussed. In section 5, details are given for the univariate interpolation of geometry and variable along each edge when cubic B-splines are used. In section 6, the development of a Coons-patch macroelement is described. In section 7, the development of large 3-D boundary elements is given for an elastic structure. In section 8, the basic idea of developing trivariate Coons macroelements is given for the first time. In section 9, the proposed methodologies are compared with conventional ones on the basis of number of nodes used. In section 10, a numerical example is given for the dynamic (acoustic) analysis of a paralleloidal cavity, where the proposed methods are numerically compared with conventional ones. In section 11, a critical comparison between the proposed trivariate Coons interpolation from one side and trivariate NURBS representation as well as other methodologies from the other side is attempted. Finally, section 12 summarizes the conclusions of this paper.

2. Coons' interpolation formula

2.1 Bivariate formula: 2-D patch

Generally, two cases of two-dimensional patches can be distinguished:

- Quadrilateral patches
- Triangular patches

2.1.1 Quadrilateral patch

A four-sided region ABCD, as shown in **Figure 2(a)**, can easily be mapped to a unit square in the *rs* parametric domain shown in **Figure 2(b)** by the method of Coons' patch [18]. Following [43], for purposes of generalization the relevant theory is given below using suitable projections.

First, the concept of the *lofting projector* '*P*' is introduced. This projector is any idempotent linear operator, which maps a true surface to an approximate surface, subject to certain interpolatory constraints.

Let us assume that the Cartesian co-ordinates $\mathbf{x}(r,s)=\{x,y,z\}^T$ in *A*, with *r* and *s* denoting normalised co-ordinates, are known at the boundaries (*r*=0,1; *s*=0,1) of a curvilinear patch of area *A*. Let us also define the well known cardinal blending functions:

$$E_0(r)=1-r\ ,\ E_1(r)=r\quad ;\quad E_0(s)=1-s\ ,\ E_1(s)=s \tag{1}$$

Now, the following unidirectional, or lofting, operators $P_r\{\mathbf{x}\}$ and $P_s\{\mathbf{x}\}$ may be constructed (summation over repeated indices is understood):

$$
\begin{aligned}
P_r\{\mathbf{x}\} &= \mathbf{x}(r_i, s) \cdot E_i(r), \\
P_s\{\mathbf{x}\} &= \mathbf{x}(r, s_j) \cdot E_i(s).
\end{aligned}
\tag{2}
$$

The above lofting operators form the basis for the definition of more complex operators with blending interpolation properties in more than one direction. So, the two-dimensional lofting operator

$$
P_{rs}\{\mathbf{x}\} = P_r P_s\{\mathbf{x}\} = \mathbf{x}(r_i, s_j) \cdot E_i(r) \cdot E_j(s) \tag{3}
$$

can be constructed with the aid of the unidirectional operators $P_r\{\mathbf{x}\}$ and $P_s\{\mathbf{x}\}$.

Finally, the co-ordinates of any point in the interior of the curvilinear patch is approximated as:

$$
\mathbf{x}(r, s) = (P_r + P_s - P_{rs})\{\mathbf{x}\} \tag{4}
$$

or, using conventional notation, as

$$
\begin{aligned}
\mathbf{x}(r, s) = (1-r)\mathbf{x}(0, s) + r\mathbf{x}(1, s) + (1-s)\mathbf{x}(r,0) + s\mathbf{x}(r,1) \\
- (1-r)(1-s)\mathbf{x}(0,0) - r(1-s)\mathbf{x}(1,0) - (1-r)s\mathbf{x}(0,1) - rs\mathbf{x}(1,1)
\end{aligned}
\tag{5}
$$

2.1.2 Triangular patch

A three-sided region can be similarly divided into a mesh of triangular elements by the use of a trilinearly blended interpolant, as described by Barnhill and coworkers [44-46]. A simple formulation may be also found in recent CAD textbooks such as [47,p.244]. The three-sided region shown in **Figure 3(a)** can be mapped to the parametric domain shown in **Figure 3(b)** by

$$P(u,v,w) = \frac{1}{2}\left[\frac{ug(v)}{1-v} + \frac{wh(1-v)}{1-v} + \frac{vh(w)}{1-w} + \frac{uf(1-w)}{1-w} + \frac{wf(u)}{1-u} + \frac{vg(1-u)}{1-u} - wf(0) - ug(0) - vh(0)\right] \tag{6}$$

The parametric domain for Equation (6) is expressed as

$$u+v+w=1 \quad 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1 \tag{7}$$

The parametric domain in this case can be sliced by incrementing *u,v* values between 0 and 1 and evaluating the corresponding *w* values from each set of *u* and *v*.

2.2 Trivariate formula: 3-D boxlike volume (superbrick)

A boxlike region ABCDEFGH, shown in **Figure 4(a)**, can easily be mapped to a unit cube in the *rts* parametric domain ($0 \le r,s,t \le 1$) shown in **Figure 4(b)**. The relevant formula may be found in [18,p.41] and, as previously mentioned, is has been applied by Cook [39] for mesh generation purposes. For more details the interested reader may also consult standard textbooks [38,48]. The only difference with a quadrilateral patch is that here six equations associated to the boundary surfaces are blended, instead of four equations associated to the boundary edges of the patch. Below, the same formula is written below in terms of projections.

Now, besides the above-mentioned $P_r\{\mathbf{x}\}, P_s\{\mathbf{x}\}$ and $P_{rs}\{\mathbf{x}\}$ operators, one-, two- and three-dimensional operators are further introduced as follows:

$$\begin{aligned}
P_t\{\mathbf{x}\} &= \mathbf{x}(r,s,t_k)\cdot E_k(t) \\
P_{st}\{\mathbf{x}\} &= P_s P_t\{\mathbf{x}\} = \mathbf{x}(r,s_j,t_k)\cdot E_j(s)\cdot E_k(t) \\
P_{rt}\{\mathbf{x}\} &= P_r P_t\{\mathbf{x}\} = \mathbf{x}(r_i,s,t_k)\cdot E_i(r)\cdot E_k(t) \\
P_{rst}\{\mathbf{x}\} &= P_r P_s P_t\{\mathbf{x}\} = \mathbf{x}(r_i,s_j,t_k)\cdot E_i(r)\cdot E_j(s)\cdot E_k(t)
\end{aligned} \tag{8}$$

Again, summation over repeated indices is understood.

Having introduced the one-, two- and three-dimensional operators, then the following formula describes the interpolation of the co-ordinate vector $\mathbf{x}(r,s,t)$:

$$\begin{aligned}\mathbf{x}(r,s,t) = &-(-1)^1 \cdot \left(P_r + P_s + P_t\right)\{\mathbf{x}\} \\ &-(-1)^2 \cdot \left(P_{rs} + P_{st} + P_{st}\right)\{\mathbf{x}\} \\ &-(-1)^3 \cdot P_{rst}\{\mathbf{x}\}.\end{aligned} \tag{9}$$

3. Equivalent expressions of trivariate Coons' formula

3.1 General remarks

Equation (9) is generally applicable but it can be further simplified and be written in more manageable expressions. So, in the case of a generalised curvilinear paralleloid (boxlike region), the geometry includes:

(a) six surfaces, S

(b) twelve edges, E, and

(c) eight corners, C

Obviously, equation (9) includes all three quantities: Surfaces (S), edges (E) and corners (C). For convenience, the projections related to the S, E and C are denoted as follows:

$$\begin{aligned}S\{\mathbf{x}\} &= \left(P_r + P_s + P_t\right)\{\mathbf{x}\} \\ E\{\mathbf{x}\} &= \left(P_{rs} + P_{st} + P_{tr}\right)\{\mathbf{x}\} \\ C\{\mathbf{x}\} &= P_{rst}\{\mathbf{x}\}.\end{aligned} \tag{10}$$

However, in the case of adequately smooth and regular surfaces, the edges E can sufficiently describe S. In fact, by applying eq.(4) on the six surfaces of the superbrick, one can easily derive the following relationship:

$$S\{\mathbf{x}\} + 3C\{\mathbf{x}\} = 2E\{\mathbf{x}\} \tag{11}$$

Then, by substituting equations (10,11) into equation (9), one can finally derive three equivalent expressions of the three-dimensional Coons' interpolation formula, as follows:

$$\mathbf{x}(r,s,t) = S\{\mathbf{x}\} - E\{\mathbf{x}\} + C\{\mathbf{x}\} \tag{12.a}$$

$$\mathbf{x}(r,s,t) = 1/2\, S\{\mathbf{x}\} - 1/2\, C\{\mathbf{x}\} \tag{12.b}$$

$$\mathbf{x}(r,s,t) = E\{\mathbf{x}\} - 2C\{\mathbf{x}\} \tag{12.c}$$

3.2 Comments on equivalent expressions [equations (12)]

Obviously, equation 12(a) is the most and general expression because it includes any type of the surrounding surfaces *S*. Moreover, Equation 12(b) is obtained by eliminating the edges (*E*) and it is based on the surface *S*, the last being corrected by the co-ordinates of the corners *C*. Finally, equation 12(c) includes only the twelve edges *E* and eight corners *C*, or in other words, the absolutely necessary data for the construction of a Coons' block made of Coons' surfaces.

In conclusion, in cases where the superbrick is sufficiently regular, equation 12(c) is the most advantageous and will be therefore applied thoroughly in this paper. Using conventional notation, equation 12(c) becomes:

$$\begin{aligned}
\mathbf{x}(r,s,t) = {} & (1-s)(1-t)\ \mathbf{x}(r,0,0) + r\,(1-s)\ \mathbf{x}(1,0,t) \\
& + (1-s)\,t\ \mathbf{x}(r,0,1) + (1-r)(1-s)\ \mathbf{x}(0,0,t) \\
& + s\,(1-t)\ \mathbf{x}(r,1,0) + r\,s\ \mathbf{x}(1,1,t) \\
& + s\,t\ \mathbf{x}(r,1,1) + (1-r)s\ \mathbf{x}(0,1,t) \\
& + (1-r)(1-t)\ \mathbf{x}(0,s,0) + (1-r)t\ \mathbf{x}(0,s,1) \\
& + r\,(1-t)\ \mathbf{x}(1,s,0) + r\,t\ \mathbf{x}(1,s,1) \\
& - 2\big[(1-r)(1-s)(1-t)\ \mathbf{x}(0,0,0) + (1-r)(1-s)t\ \mathbf{x}(0,0,1) \\
& + (1-r)s\,(1-t)\ \mathbf{x}(0,1,0) + (1-r)s\,t\ \mathbf{x}(0,1,1) \\
& + r(1-s)(1-t)\mathbf{x}(1,0,0) + r(1-s)t\,\mathbf{x}(1,0,1) \\
& + r\,(1-s)(1-t)\ \mathbf{x}(1,1,0) + r\,s\,t\ \mathbf{x}(1,1,1)\ \big]
\end{aligned} \tag{13}$$

4. Mesh generation

Although both bivariate and trivariate Coons interpolation formulas (equations (5) and (13)) have been used in the past for mesh generation purposes [38], the quality of the produced meshes is usually very poor and requires a smoothening. This paper presents a simple a-posteriori technique to smoothen the initial mesh for both two- and three-dimensional cases.

4.1 Two-dimensional curvilinear surface patch

The necessary steps are as follows:

(1) A logical square ABCD of dimensions 1×1, called reference square, is considered (**Figure 2b**). Each of its four sides corresponds to a part of the real boundary of the curvilinear patch. Normalised boundary co-ordinates (r,s) are calculated.

(2) The co-ordinates of internal points are determined by applying equation (5).

(3) Node numbers and element connectivity are generated.

(4) The co-ordinates of internal points are updated applying the following equation:

$$\mathbf{x}_{new} = \frac{\sum_{j=1}^{8} \left(\mathbf{x}_{old}\right)_j}{8} \tag{14}$$

for a few times. In equation (14), the co-ordinates of each internal point are updated on the basis of the eight surrounding nodes (north, south, east, west and corners). It is also possible to change the denominator in eq.(14) from 8 to 4 by excluding the corners. Alternatively, different weighting factors might be used in both cases. The latter resembles to a technique well-known in the common praxis of the finite-difference method during the solution of *Laplace* equation. In this sense, the so-produced mesh resembles to the result obtained using an elliptic mesh generation.

As an example, for the component shown in **Figure 5**, the direct application of eq.(5) leads to the mesh shown in **Figure 6(a)** while after only three iterations using eq.(14) the mesh improves to that shown in **Figure 6(b)**.

4.2 Three-dimensional superbrick

For three-dimensional meshes where a logical cube ABCDEFGH of dimensions 1×1×1 is considered (**Figure 4b**), instead of a logical square, we proceed as follows:

(1) For each surface of the cube the generated mesh nodes are determined as explained previously.

(2) Mesh nodes inside the cube are determined by the three-dimensional version of the Coons' formula (eq.13)

(3) Co-ordinates of internal points are updated using the following equation:

$$\mathbf{x}_{new} = \frac{\sum_{j=1}^{8} \left(\mathbf{x}_{old}\right)_j}{26} \tag{15}$$

for a few times. In equation (15), the co-ordinates of each internal point are updated on the basis of the twenty-six closest nodes (front, back, up, down and corners).

As previously, the finally produced mesh is regular and lines (with *r,s,t*=const) tend to become perpendicular each other. Here, it is also possible to exclude the corner nodes and reduce the denominator from 26 to 6. In this case, the proposed procedure resembles to a relaxation method for the solution of Laplace equation within the solid patch, under Dirichlet boundary conditions. The latter conditions are merely the prescribed co-ordinates. This fact classifies the proposed mesh generator in the category of elliptic mesh generators.

As a test case, we choose the superbrick shown in **Figure 7**; this is more complicated than a usual curved paralleloid. Clearly, it can be noticed that the edges DA and HE are not simple but each of them consists of three line segments. The rough and smoothened meshes using eq.(15) are shown in **Figures 8(a)** and **8(b)**, respectively.

4.3 Field of applications

The proposed smoothening algorithms are useful for the following cases:

I) The combination of equations (5,14) is useful for mesh generation of (a) plates and shells as well as (b) plane-stress or plane-strain structures, when the FEM is applied. This is also useful for meshing the boundaries of the solid structure for BEM analysis using conventional elements.

II) The combination of equations (13,15) is useful for the FEM as well as the FDM (finite difference) or FVM (finite volume) analysis within boxlike regions.

5. Univariate interpolation

In all following cases, either two- or three-dimensional, the key-point is to properly interpolate the unknown quantity (potential or displacement) along the lines (sides or edges: AB, BC, etc.) of the boundary. This can be achieved using any reasonable set of interpolation functions such as Lagrange polynomials, piecewise linear [28,29], piecewise quadratic, cubic B-splines [6,29], et cetera.

In general, having prescribed q (different) nodes along a side (or edge), for example $u(\xi_i)$, i=1,2,…, q, an appropriate interpolating formula for the function $u(\xi)$

is sought. Considering that q may be allowed to be a large number, a Lagrangian interpolation polynomial would tend to produce undesirable oscillations between two arbitrary abscissae ξ_i and ξ_{i+1}, as it may possess as many as $(q-1)$ maxima and minima over its entire interval of variation. For this reason, the use of splines is envisaged. So, given q degrees of freedom on the boundary of the patch at $\xi_1, \xi_2, ..., \xi_q$, a spline function $B(\xi)$ of degree m is a function having the two following properties [49]:

(1) In each interval (ξ_i, ξ_{i+1}), $i = 1,2,...,q-1$, $B(\xi)$ is given by a polynomial of degree m or less.

(2) $B(\xi)$ and its derivatives of order 1, 2, …, m-1 are continuous everywhere.

A commonly used spline function is the truncated power function $\langle \xi - \xi_i \rangle^m$, for any variable $\xi - \xi_i$ and for any positive integer m. This function is defined by:

$$\begin{aligned} \langle \xi - \xi_i \rangle^m &= (\xi - \xi_i)^m \quad , \quad for\ \xi - \xi_i \rangle 0; \\ \langle \xi - \xi_i \rangle^m &= 0 \qquad\qquad , \quad for\ \xi - \xi_i \langle 0 \end{aligned} \tag{16}$$

It is easily seen that the function $B(\xi)$ has a unique representation of the form [56]:

$$B(\xi) = b_0 + b_1\xi + b_2\xi^2 + \ldots + b_{m-1}\xi^{m-1} + \sum_{i=1}^{q-1} a_i \langle \xi - \xi_i \rangle^m = P(\xi) + \sum_{i=1}^{q-1} a_i \langle \xi - \xi_i \rangle^m \tag{17}$$

with $P(\xi)$ denoting a polynomial of degree (m-1) and a_i properly chosen constants. The most common case is that the spline of degree m = 3 (order m+1=4), that is of cubic B-splines. If now $B_i(\xi)$ denote *cardinal* splines of degree m, then the function $u(\xi)$ could be written in the following form:

$$u(\xi)=\sum_{j=1}^{q} B_j(\xi) u(\xi_j) \tag{18}$$

6. Development of two-dimensional Coons-patch macroelements

So far, relevant macroelements have been developed for potential [6,22-24,27,28] and elasticity problems [20,21,25,26,29], both static [20,21,24,27-29] and dynamics [20,22-27]. The general procedure is as follows:

Each macro-element is considered to occupy the entire four-sided curvilinear patch ABCD shown in Figure 2(a). The variable, *u*, along each of the four boundary sides AB, BC, CD and DA can now be expressed using interpolations similar to that of equation (18). It should be here clarified that it is not necessary that Bj is a cubic B-spline but it could be any other typical basis functions such as piecewise polynomial and similar.

By arranging q_1, q_2, q_3 and q_4 nodal points along the sides AB, BC, CD and DA, respectively, the total number of the nodal points becomes:

$$q_e = q_1 + q_2 + q_3 + q_4 - 4 \tag{19}$$

After the co-ordinates of the boundary nodes have been normalized, then a mapping between the real patch ABCD and a unit square $\mathrm{A'B'C'D'}$ (Figure 2(b)) can be established. In virtue of equation (5), for a given couple (*r,s*) in the interior of $\mathrm{A'B'C'D'}$, the cartesian co-ordinates $\mathbf{x}(r,s)=\{x(r,s),\ y(r,s)\}^{\mathrm{T}}$ of the corresponding point along the real patch ABCD, as well as the unknown variable *u*(*r,s*), can be approximated on the basis of its four given geometrical boundaries.

The next step is to assume that, equation (5) does not only interpolate the geometry, but also the unknown the variable, *u*. In other words, we extend the idea of *isoparametric* elements [50] from small to large size. Then, we have to interpolate the univariate boundary displacements $u(r,0)$, $u(1,s)$, $u(r,1)$ and $u(0,s)$ in equation (5) using a suitable set of trial functions, as in equation (18).

By applying equation (18) on the four sides of the patch ABCD, one obtains:

$$\begin{aligned} u(r,0) &= \sum_{j=1}^{q_1} B_j^{(1)}(r)u(r_j,0), \quad & u(1,s) &= \sum_{j=1}^{q_2} B_j^{(2)}(s)u(1,s_j) \\ u(r,1) &= \sum_{j=1}^{q_3} B_j^{(3)}(r)u(r_j,1), \quad & u(0,s) &= \sum_{j=1}^{q_4} B_j^{(4)}(s)u(0,s_j) \end{aligned} \tag{20}$$

By substituting eqs.(20) into eq.(5), applicable for the variable u, it is easily found that the unknown variable $u(r,s)$ inside the reference macro-element is approximated as follows :

$$u(r,s) = \sum_{j=1}^{q_e} N_j(r,s)\, u_j \ , \tag{21}$$

Using the global shape functions N_j appearing in equation (21), which is applicable to degrees of freedom along the boundary only, we can write the well-known finite element expression for the general case of dynamic analysis [50]:

$$[\mathbf{M}]\cdot\{\ddot{\mathbf{u}}(t)\} + [\mathbf{K}]\cdot\{\mathbf{u}(t)\} = \{\mathbf{f}(t)\} \tag{22}$$

In equation (22), [**M**] denotes the mass matrix, [**K**] the stiffness matrix, {**f**} the imposed external forces (or “fluxes”) and {**u**} the resulting displacements (or potentials). Both [**M**] and [**K**] are symmetric and fully occupied. In dynamic problems both [**M**] and [**K**] matrices participate (plus damping) while in static problems only [**K**]. For more details, the interested reader may consult a recent publication [29]. Typical illustrations of cardinal global shape functions may also found in [27,30]. Finally, it is clarified that in eq.(22), the variable t denotes the time and this symbol is irrelevant to the third normalized coordinate met in equations (8)-(13).

7. Generation of large three-dimensional boundary elements

This methodology is closely related to the work of Casale [3-5] but it was independently developed in early 1990s. However, there are some differences. Casale [3-5] uses *virtual nodes* and combines his method with Lagrange interpolation functions. On the contrary, we use nodal variables distributed along the boundaries of the patches; no unknown degree of freedom is considered within each patch. Moreover, instead of Lagrange polynomials [3-5], that are prone to numerical oscillations in case of many nodes, we use cubic B-splines that lead to a smooth interpolation of the boundary data (displacements/potentials and tractions/fluxes).

In both methods, the boundary of the domain is divided into a small number of large trimmed patches. For boxlike regions (superbricks), the number of patches may be for example equal to six but there is no restriction for that. In virtue of equation (21), the unknown variable, u, along each patch can be expressed in terms of global shape functions and degrees of freedom arranged along the four (or three) sides of the relevant patch. Besides, the same interpolation holds for the primary variable (potential or displacement, u) as well as its derivative (flux or traction, p).

In the case of elasticity problems, the co-ordinate vector within the l-th patch, possessing q_l nodes, is interpolated on the basis of the boundaries of the patch as follows:

$$\mathbf{x}(r,s) = \sum_{j=1}^{q_l} N_j(r,s)\,\mathbf{x}_i^j = \mathbf{N}\,\mathbf{x} \tag{23}$$

Since each patch is considered as an isoparametric element it holds that:

$$\begin{aligned} \mathbf{u}(r,s) &= \sum_{j=1}^{q_l} N_j(r,s)\,\mathbf{u}_i^j = \mathbf{N}\,\mathbf{u} \\ \mathbf{p}(r,s) &= \sum_{j=1}^{q_l} N_j(r,s)\,\mathbf{p}_i^j = \mathbf{N}\,\mathbf{p} \end{aligned} \tag{24}$$

By substituting Eq. (24) in the usual integral equation (see for example [51]) and summarizing over the N_p patches in which the boundary is divided, one obtains:

$$
\begin{aligned}
& c^i \mathbf{u}^i + \sum_{ip=1}^{Np} \left\{ \oiint_{\Gamma_{ip}} \mathbf{p}^*_{lk} \mathbf{N}_k \, \mathrm{d}\Gamma \right\} \mathbf{u}_{ip} \\
& \quad = \sum_{ip=1}^{Np} \left\{ \oiint_{\Gamma_{ip}} \mathbf{u}^*_{lk} \mathbf{N}_k \, \mathrm{d}\Gamma \right\} \mathbf{p}_{ip}
\end{aligned}
\tag{25}
$$

In Equation (25) the infinitesimal area $\mathrm{d}\Gamma$ is given by:

$$
d\Gamma = \left| G(r,s) \right| \mathrm{d}r \, \mathrm{d}s \tag{26}
$$

where the Jacobian is calculated as:

$$
\begin{aligned}
& \left| G(r,s) \right| = \left(g_1^2 + g_2^2 + g_3^2 \right)^{1/2} \\
& g_1 = \frac{\partial x_2}{\partial r} \frac{\partial x_3}{\partial s} - \frac{\partial x_2}{\partial s} \frac{\partial x_3}{\partial r} \\
& g_2 = \frac{\partial x_1}{\partial r} \frac{\partial x_3}{\partial s} - \frac{\partial x_1}{\partial s} \frac{\partial x_3}{\partial r} \\
& g_3 = \frac{\partial x_1}{\partial r} \frac{\partial x_2}{\partial s} - \frac{\partial x_1}{\partial s} \frac{\partial x_2}{\partial r}
\end{aligned}
\tag{27}
$$

Now, for the purposes of the numerical integration only, the patch is divided into $N_r \times N_s$ cells where a second set of normalized co-ordinates $\left(-1 \le r', s' \le 1\right)$ is introduced [30]. Therefore, the term $\left|G(r,s)\right| dr\, ds$ in Eq. (26) is replaced by $\left|G(r,s) \cdot G'(r',s')\right| dr'\, ds'$, which requires a trivial (e.g., 2×2, 3×3, 4×4) Gaussian quadrature. A selective integration scheme has been recently developed.

Therefore, the final algebraic system obtains the form:

$$
\mathbf{C}\,\mathbf{U} + \sum_{ip=1}^{Np} \hat{\mathbf{H}}^{ip}\, \mathbf{U}^{ip} = \sum_{ip=1}^{Np} \mathbf{G}^{ip}\, \mathbf{P}^{ip} \tag{28}
$$

where $\mathbf{U}$ is the displacement vector of all nodes on the boundary of the structure (along the patch edges), $\mathbf{U}^{ip}$ and $\mathbf{P}^{ip}$ are displacement and traction vectors referring to the *ip*-th patch as well as $\mathbf{H}^{ip}$ and $\mathbf{G}^{ip}$ are the nonsymmetric influence matrices.

By properly assembling the submatrices in eq.(28) we obtain:

$$\mathbf{H}\,\mathbf{U} = \mathbf{G}\,\mathbf{P} \tag{29}$$

If q is the total number of nodes along all edges of the patches on the boundary of the structure, the dimensions of the vectors and matrices in eq.(29) are as follows:

U : displacement vector ($3q \times 1$)

P : traction vector ($3\overline{q} \times 1$)

H : total displacement –influence matrix ($3q \times 3\,q$)

G : total traction-influence matrix ($3q \times 3\,\overline{q}$)

The above symbol $\overline{q}$ is larger than q ($\overline{q} = q + \Delta q$) with Δq depending on the number of sharp corners and their multiplicity in traction discontinuity [51].

The above static analysis described by eq.(29) can be also extended to the solution of dynamic problems too. Briefly, using a set of radial basis functions a mass matrix is constructed for the entire structure and it is combined with the static matrices **H** and **G** shown in eq.(29) [61,62].

8. Development of trivariate Coons macroelements (large 3-D finite elements)

Let us consider a three-dimensional boxlike region (superbrick) *V*, which is entirely occupied by the 3D- macroelement. Two co-ordinate systems shown in **Figure 4**, a Cartesian (*x, y, z*) and a body-oriented (*r, s, t*) with $0 \le r,s,t \le 1$ are distinguished. Thus, *s*=0 and *t*=0 along the boundary edge designated 1 in **Figure 4**. Similarly, *r*=1 and *s*=0 along the boundary edge designated 2; and in this manner all twelve edges may be defined.

Define $\mathbf{x}(r,0,0) = [x(r,0,0), y(r,0,0), z(r,0,0)]^T$ as the boundary edge functions that specify x, y and z along the boundary edge designated 1. In a similar manner, other boundary functions can be defined to specify x, y and z along the rest eleven boundary edges, as summarized below:

Boundary edge

1=AB $(s=0, t=0)$ $\mathbf{x}_{AB} = \mathbf{x}(r,0,0) = [x(r,0,0), y(r,0,0), z(r,0,0)]^T$ (30a)

2=BC $(r=1, s=0)$ $\mathbf{x}_{BC} = \mathbf{x}(1,0,t) = [x(1,0,t), y(1,0,t), z(1,0,t)]^T$ (30b)

3=CD $(s=0, t=1)$ $\mathbf{x}_{CD} = \mathbf{x}(r,0,1) = [x(r,0,1), y(r,0,1), z(r,0,1)]^T$ (30c)

4=DA $(r=0, s=0)$ $\mathbf{x}_{DA} = \mathbf{x}(0,0,t) = [x(0,0,t), y(0,0,t), z(0,0,t)]^T$ (30d)

5=BF $(r=1, t=0)$ $\mathbf{x}_{BF} = \mathbf{x}(1,s,0) = [x(1,s,0), y(1,s,0), z(1,s,0)]^T$ (30e)

6=CG $(r=1, t=1)$ $\mathbf{x}_{CG} = \mathbf{x}(1,s,1) = [x(1,s,1), y(1,s,1), z(1,s,1)]^T$ (30f)

7=DH $(r=0, t=1)$ $\mathbf{x}_{DH} = \mathbf{x}(0,s,1) = [x(0,s,1), y(0,s,1), z(0,s,1)]^T$ (30g)

8=AE $(r=0, t=0)$ $\mathbf{x}_{AE} = \mathbf{x}(0,s,0) = [x(0,s,0), y(0,s,0), z(0,s,0)]^T$ (30h)

9=EF $(s=1, t=0)$ $\mathbf{x}_{EF} = \mathbf{x}(r,1,0) = [x(r,1,0), y(r,1,0), z(r,1,0)]^T$ (30i)

10=FG $(r=1, s=1)$ $\mathbf{x}_{FG} = \mathbf{x}(1,1,t) = [x(1,1,t), y(1,1,t), z(1,1,t)]^T$ (30j)

11=GH $(s=1, t=1)$ $\mathbf{x}_{GH} = \mathbf{x}(r,1,1) = [x(r,1,1), y(r,1,1), z(r,1,1)]^T$ (30k)

12=HE $(r=0, s=1)$ $\mathbf{x}_{HE} = \mathbf{x}(0,1,t) = [x(0,1,t), y(0,1,t), z(0,1,t)]^T$ (30l)

Again, the keypoint is to assume that eq.(13) holds for both the geometry representation, **x**, and the unknown variable, u, within the volume V. In other words, the superbrick is here considered as a large *isoparametric* element.

By properly approximating the variable u along the twelve edges (using, for example, splines, piecewise linear interpolation, Lagrange polynomials, etc), in virtue of equation (18) the following set is derived:

$$\begin{aligned} u(r,l,m) &= \sum_{j=1}^{q_r} B_j(r)\cdot u(r_j,l,m) \\ u(l,s,m) &= \sum_{j=1}^{q_s} B_j(s)\cdot u(l,s_j,m) \\ u(l,m,t) &= \sum_{j=1}^{q_t} B_j(t)\cdot u(l,m,t_j) \end{aligned} \tag{31}$$

In eq.(31), q_r, q_s and q_t denote the number of nodes along the twelve edges being parallel to the r, s and t local axes, respectively. Obviously, it is not necessary to have the same number of nodes even along parallel edges to either of r, s and t.

Then, by substituting equations (31) into equation (13) one can obtain the global functions:

$$u(x,y,z) = \sum_{k=1}^{q} N_k\left[x(r,s,t), y(r,s,t), z(r,s,t)\right]\cdot u_k \tag{32}$$

In equations (32), $N_k(x,y,z)$ denotes the *global* three-dimensional cardinal shape functions and u_k the nodal degrees of freedom appearing only at the twelve boundary edges of the 3D-macroelement. Obviously, by taking into consideration that every number of the set (q_1, q_2,…, q_{12}) includes both of this ends, the total number of nodal points of the macroelement is given by

$$q = \sum_{i=1}^{12} q_i - 16 \tag{33}$$

Using the global shape functions appearing in equation (32), one can derive a similar matrix equation of motion with that of equation (22). As previously, mass and stiffness matrices are symmetric and fully occupied.

9. Comparison between several methods

9.1 Two-dimensional problem

Let us assume that the patch is divided into n_1 and n_2 segments per direction. Therefore, by reserving the same boundary nodes, the total number of them are as follows:

FEM solution : $(n_1 + 1)(n_2 + 1)$

BEM solution : $2(n_1 + n_2)$

Bivariate-Coons-patch: $2(n_1 + n_2)$

Obviously, the proposed two-dimensional Coons-patch macroelement has the same number of nodes with the conventional BEM solution. However, its advantage is that (i) it has symmetric mass and stiffness matrices, and (ii) in dynamic problems it behaves much well than BEM as it has been clearly shown in [6,27], among others. Moreover, as it essentially reserves the FEM formulation, it can be easily linked into any FEM code and couple with conventional finite elements.

9.2 Three-dimensional problem

Let us assume that the superbrick is divided into n_1, n_2 and n_3 segments per direction. Therefore, the corresponding total number of nodes is as follows:

FEM solution : $(n_1 + 1)(n_2 + 1)(n_3 + 1)$

BEM solution : $2(n_1n_2 + n_2n_3 + n_3n_1 + 1)$

BEM trimmed-patch : $4(n_1 + n_2 + n_3 - 1)$

Trivariate-Coons-superbrick : $2(n_1n_2 + n_2n_3 + n_3n_1 + 1)$

Again, the proposed trivariate-Coons-superbrick macroelement has the same number with the BEM trimmed-patch but it has all the advantages mentioned previously for the 2-D case.

10. Numerical example

A three-dimensional problem will be studied using all conventional methods as well as BEM-trimmed-patch and trivariate-Coons macroelements. The problem concerns a boxlike acoustical cavity of dimensions 2.5×1.1×1.0m and sound velocity c=340m/sec. The acoustical cavity is defined in three ways. In all of them a uniform discretization is chosen using ten and five subdivisions towards the directions of long and short size, respectively. First, conventional FEM solution is performed using the mesh shown in **Figure 9**. Second, the inner nodes are ignored and a conventional BEM solution is derived. Third, only the twelve edges are discretized and the relevant discretization is shown in **Figure 10**. A comparison between the number of nodes used in the several models is given in **Table 1**. Finally, as shown in **Table 2**, the accuracy of the proposed three-dimensional Coons macroelement (superbrick) is superior to BEM macroelements (using conical radial basis [61]), to conventional boundary elements as well as to conventional finite elements. It is also noted that both BEM formulations were sensitive to the selection of the constant C included into the conical radial basis, $f_j(x,y,z) = C - r$ [61]. The BEM-results in Table 2 were obtained using C≅2.91 (equals to the diagonal of the rectangular cavity) as it is suggested by Banerjee et al. [61]. In this case, the equivalent "mass" deviated from the real one by only 0.12%. However, when the constant C was chosen equal to zero, the deviation in mass increased to –2.30% (lack of mass!) and the error in the first nonzero calculated eigenfrequency changed from –0.90% to +2.92%. Moreover, it was found that there are critical values of the C-constant where the BEM formulation degenerates so that the first nonzero eigenvalue is lost.

11. Discussion

In this section we will try to comment on the several methods presented here. The general idea of this paper was to deal with the same quantities in both CAD and CAE. Concerning static analysis, it would be generally sufficient to remain on the trimmed-patch BEM solution [3-5] (perhaps updated using cubic B-splines [30-33]) but the boundary element techniques have shortcomings in dynamic problems [40-42]. Another shortcoming of the BEM is that it has some special difficulties at sharp corners and also it requires different numbering for geometry and traction nodes [51]. All these shortcomings as well as the need to be able of coupling with other conventional elements and preserving the FEM formulation was the motivation to develop the trivariate Coons-macroelements, which were presented here for the first time. In the future, the proposed method should be probably compared with other promising techniques such as finite element methods with uniform B-splines [60].

To close this section, it should be noted that there is a significant difference between the proposed trivariate Coons-macroelements and the NURBS techniques applied by other authors [14,15,17]. As it is reminded in the **Appendix A**, the univariate and bivariate B-spline functions do not equal to the unity at intermediate nodes along a segment. The same holds for interior points. For example, the reader could consult typical illustrations cited in the classical textbooks of Faux and Pratt

[55,p.179] and that of Piegl and Tiller [52, Ch.4]. As a result, the specified boundary conditions will be distributed to control points influencing the point under consideration [17]. As it was also mentioned in the introduction, this shortcoming led to the application of penalty [14,15] and Lagrange-multiplier [17] techniques. Moreover, the previous techniques [14,15,17] deal with internal points in a mesh-free approach. On the contrary, the proposed trivariate-Coons macroelements deal with nodal points directly arranged along the characteristic edges of a superbrick only and, therefore, it can apply the boundary conditions in the usual direct way. Nevertheless, a shortcoming of the proposed method might be its incapability to adapt with local control of the surfaces (excepting the case of using bubble functions [19]). However, even in case of using conventional finite elements the computational mesh usually violates in some degree the geometric model.

12. Conclusions

It was reminded that bivariate and trivariate Coons interpolation offer the mathematical background for automatic mesh generation useful for BEM and FEM analyses. In addition to that, simple smoothening procedures were here proposed. Then, it was shown that Coons interpolation offers the basis to develop large macroelements for both two- and three-dimensional problems. The advantage of the proposed trivariate Coons macroelements is that they operate *directly* on the solid modeling representation. In this way, data transfer and similar compatibility problems are minimized and data preparation costs are drastically reduced. On the other hand, shape optimization through manual or automatic procedures becomes easier because the analyzer has to deal with a smaller number of parameters. In this sense, on several levels applied, the Coons interpolation achieves to integrate CAD and CAE techniques in a unique software environment useful for the engineering design.

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APPENDIX A

Trivariate NURBS interpolation

Let us consider a three-dimensional rectangular space in the form of a deformed paralleloid (here called *superbrick*). A non-uniform rational B-spline (NURBS) volume of order n_1 in the *u* direction, n_2 in the *v* direction and n_3 in the *w* direction, is a three-dimensional trivariate vector-valued piecewise rational function of the form [53,54]:

$$\mathbf{B}(r,s,t) = \frac{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} w_{ijk}\,\mathbf{B}_{ijk}\,N_i^{n_1}(r)N_j^{n_2}(s)N_k^{n_3}(t)}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} w_{ijk}\,N_i^{n_1}(r)N_j^{n_2}(s)N_k^{n_3}(t)} \tag{A-1}$$

The $\mathbf{B}_{ijk} = (x_{ijk}, y_{ijk}, z_{ijk}) \subset \Re^3$ denote the tridirectional control points net, the $\{w_{ijk}\}$ are the weights, and the $\{N_i^{n_1}(r)\}$, $\{N_i^{n_2}(s)\}$, and $\{N_i^{n_3}(t)\}$ are the normalized B-spline basis functions defined on the knot vectors.

Similarly, equation (A-1) can be applied to the attribute model concerning the field variable (potential, acoustic pressure, displacement, etc.) as follows:

$$\mathbf{U}(r,s,t) = \frac{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} w_{ijk}\,\mathbf{U}_{ijk}\,N_i^{n_1}(r)N_j^{n_2}(s)N_k^{n_3}(t)}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} w_{ijk}\,N_i^{n_1}(r)N_j^{n_2}(s)N_k^{n_3}(t)} \tag{A-2}$$

In the most general case described by equation (A-2), the variable can be written as:

$$\mathbf{U}(r,s,t) = \sum_{I=1}^{q_e} N_I(r,s,t)\mathbf{U}_I \tag{A-3}$$

where $N_I(r,s,t)$ denote the corresponding global shape functions which are given by:

$$N_I(r,s,t) = \frac{N_i^{n_1}(r)N_j^{n_2}(s)N_k^{n_3}(t)w_{ijk}}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} N_i^{n_1}(r)N_j^{n_2}(s)N_k^{n_3}(t)w_{ijk}} \tag{A-4}$$

It is trivial to show that for each point (*r,s,t*) inside the trivariate superbrick it holds:

$$\sum_{I=1}^{q_e} N_I(r,s,t) = 1 \tag{A-5}$$

In other words, equation (A-3) seems to be similar to equation (32) and also equation (A-5) holds for both of them. However, there is an essential difference between them. For a better understanding, we assume that the weighting values have unit values, as well as we assume *cardinal* univariate functions, i.e.:

$$w_{ijk} = 1, \quad \sum_{i=0}^{n_1} N_i^{n_1}(r) = \sum_{j=0}^{n_2} N_j^{n_2}(s) = \sum_{k=0}^{n_3} N_k^{n_3}(t) = 1 \tag{A-6}$$

Under these circumstances, the rational form of the attribute model is reduced to the simpler non-rational form [1]:

$$\mathbf{U}(r,s,t) = \sum\nolimits_{i=0}^{n_1} \sum\nolimits_{j=0}^{n_2} \sum\nolimits_{k=0}^{n_3} \mathbf{U}_{ijk} N_i^{n_1}(r) N_j^{n_2}(s) N_k^{n_3}(t) \tag{A-7}$$

whence the inherent global shape function is given by:

$$N_I(r,s,t) = N_i^{n_1}(r) N_j^{n_2}(s) N_k^{n_3}(t) \tag{A-8}$$

In other words, NURBS formulation leads to global shape functions being tensor products in the form of equation (A-8). However, since the univariate functions involved in equation (A-8) do not equal to unity when applied to the corresponding nodes (Faux and Pratt [55,p.179], Piegl and Tiller [52, Ch.4]) the same holds for the global shape function N_I.

LIST OF FIGURES

LIST OF TABLES

Table 1

Formulation:	Trivariate Coons macroelement	Conventional FEM	Trimmed-patch BEM	Conventional BEM
Number of nodes	76	396	76	252

Table 2

Mode	Exact Eigen-frequencies [Hz]	Errors of calculated eigenfrequencies in %			
		Trivariate Coons macroelement	FEM Conventional	BEM Trimmed-patch	BEM Conventional
1	0.0	0.00	0.00	0.00	0.00
2	68.0	+0.02	+0.41	-0.02	-0.90
3	136.0	+0.08	+1.65	+4.48	-1.27
4	154.5	+0.16	+1.65	+6.40	-2.70
5	168.8	+0.28	+1.45	+4.32	-5.88
6	170.0	+0.16	+1.65	+6.91	-2.72
7	183.1	+0.28	+1.48	+5.08	-5.99
8	204.0	+0.15	+2.59	+8.23	-1.29

Bivariate Coons interpolation

- 2-D mesh generation
- 2-D finite macroelements
- 3-D boundary macroelements

Trivariate Coons interpolation

- 3-D FEM mesh generation
- 3-D finite macroelements

Figure 1

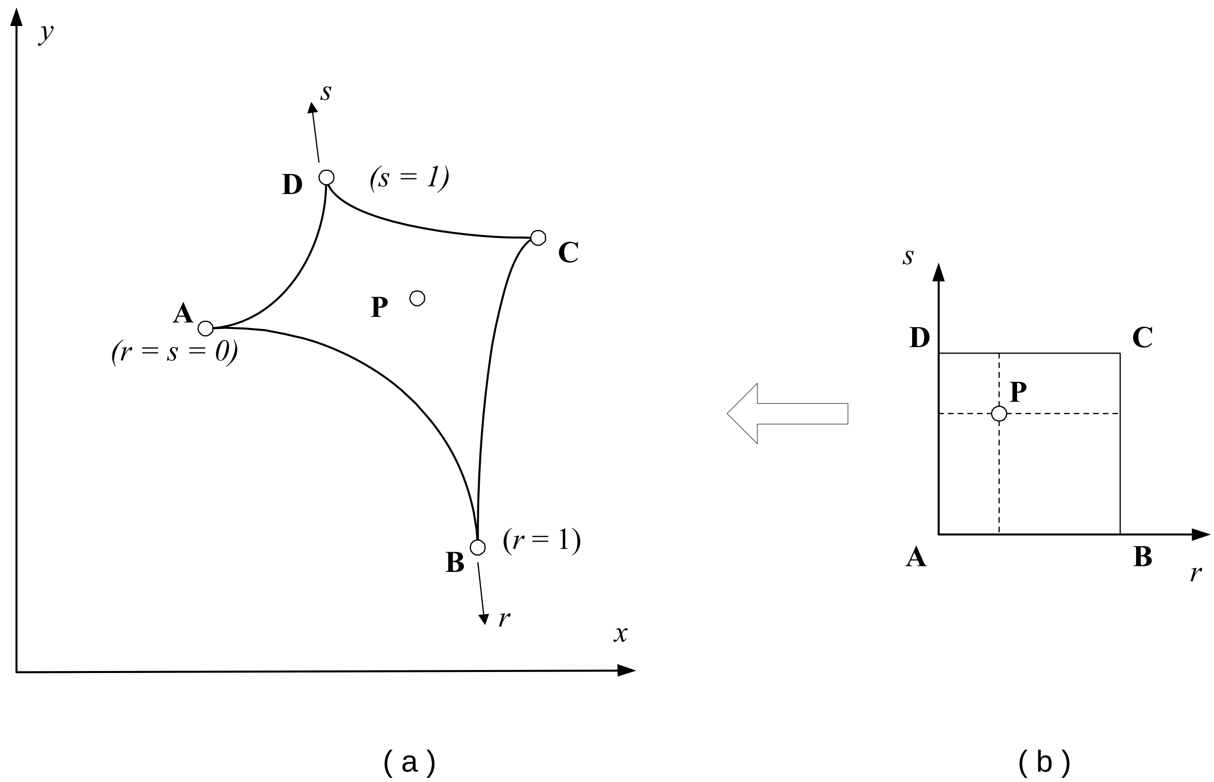

Figure 2

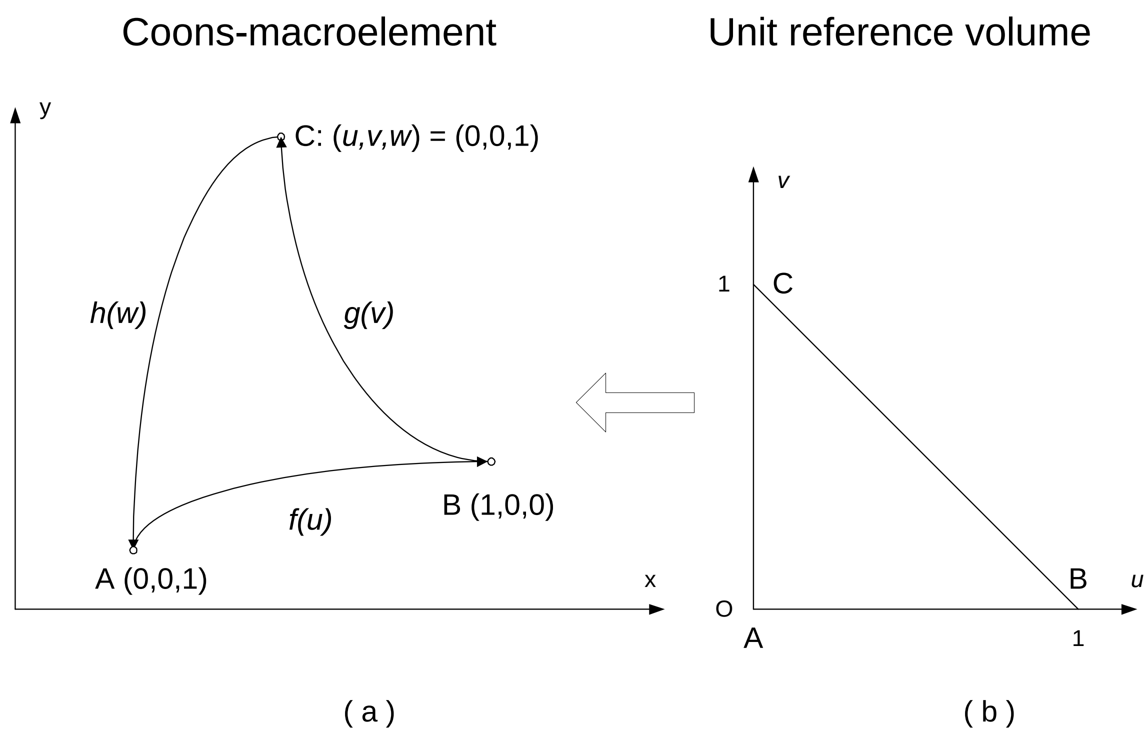
Coons-macroelement
Unit reference volume
y
C: (u,v,w) = (0,0,1)
h(w)
g(v)
f(u)
B (1,0,0)
A (0,0,1)
x
v
1
C
O
A
B
u
1
(a)
(b)

Figure 3

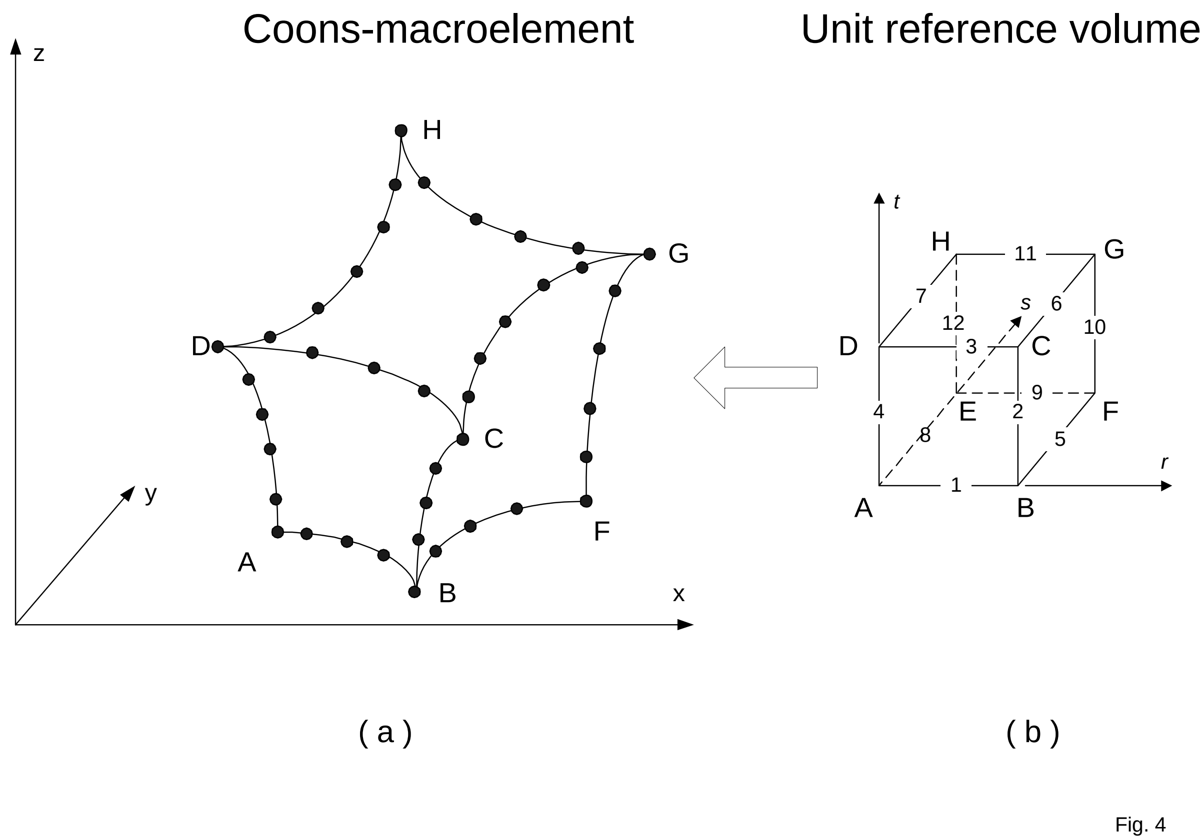
Coons-macroelement
Unit reference volume
z
y
x
H
G
D
C
F
A
B
t
s
r
11
7
6
12
3
10
9
4
8
E
2
5
1
(a)
(b)
Fig. 4

Figure 4

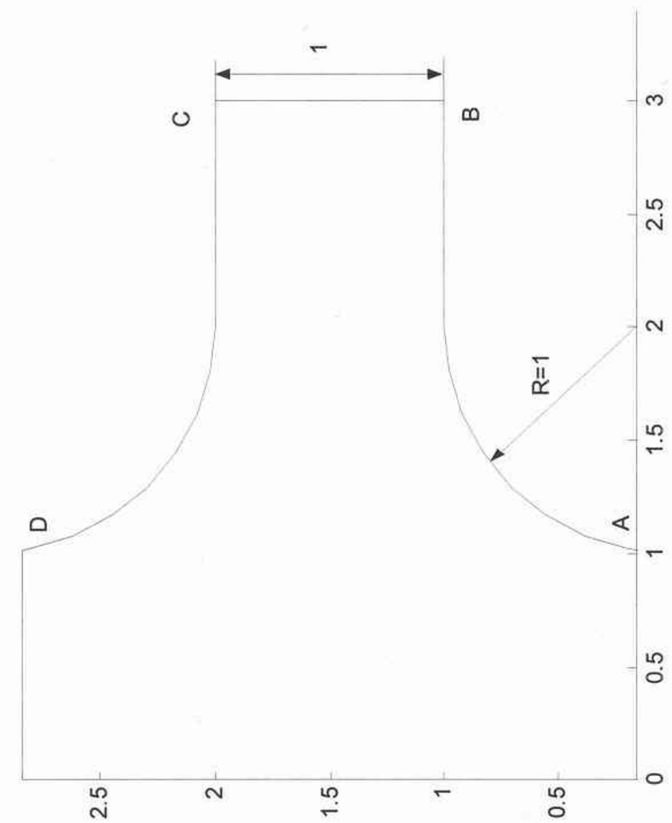

Figure 5

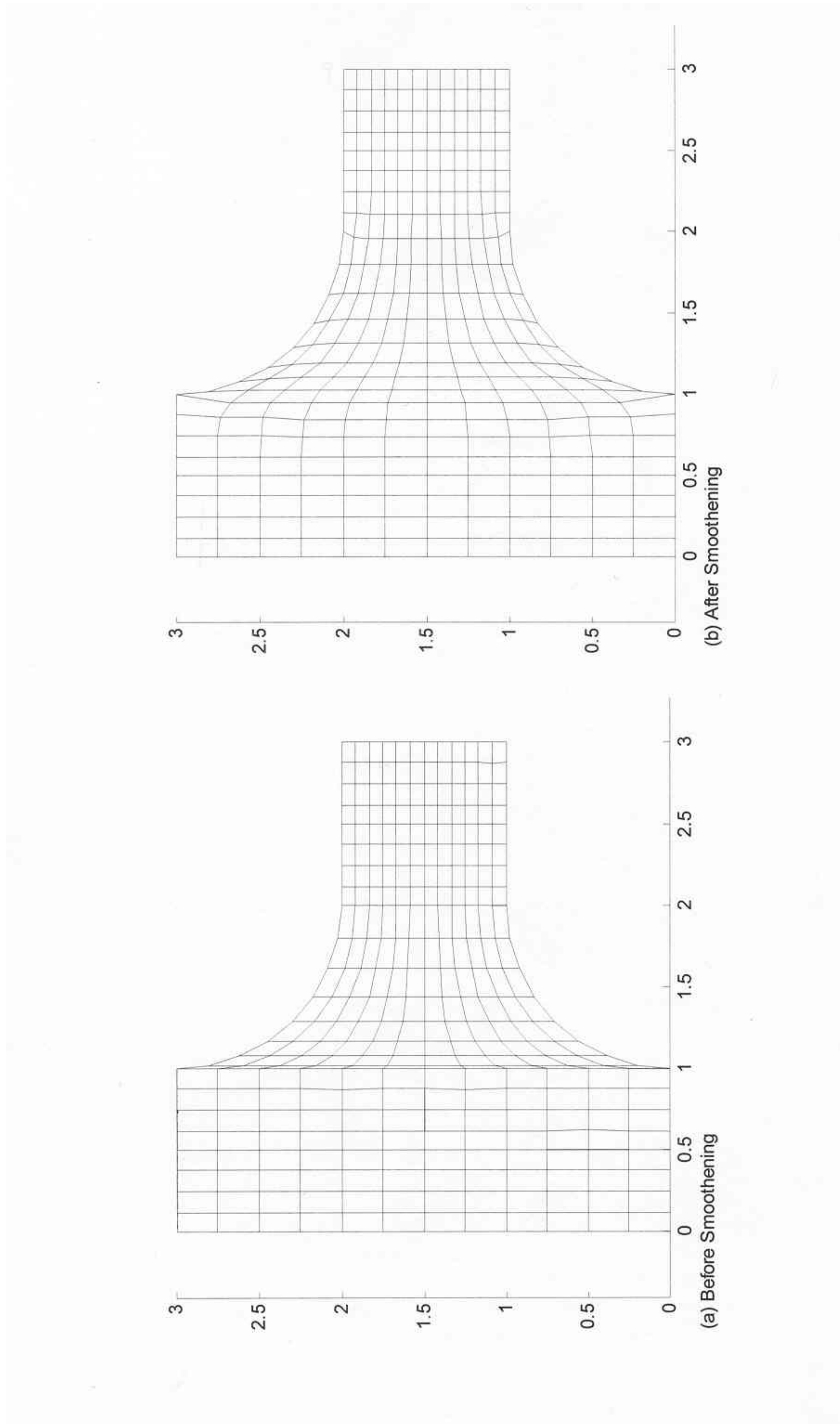

Figure 6

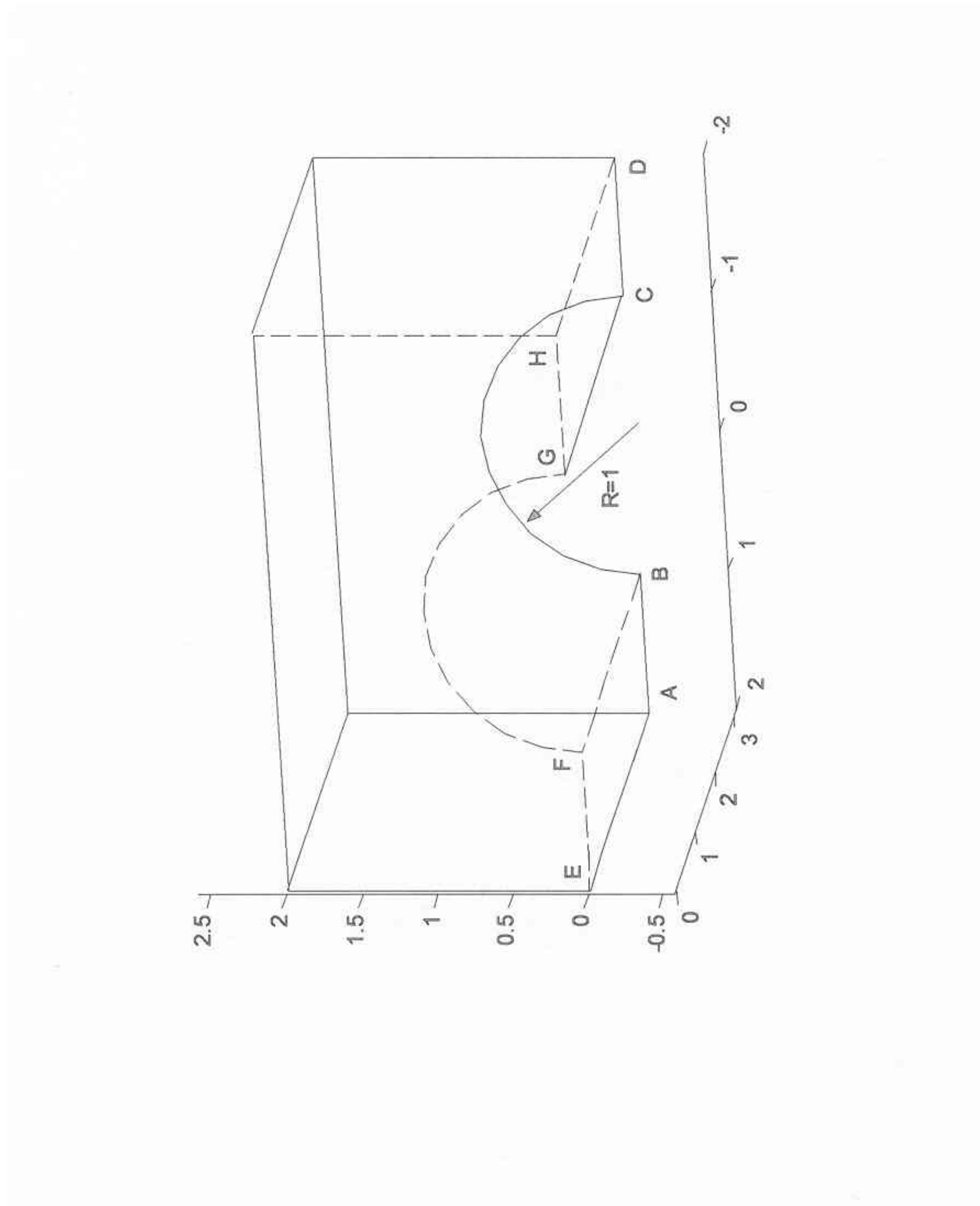

Figure 7

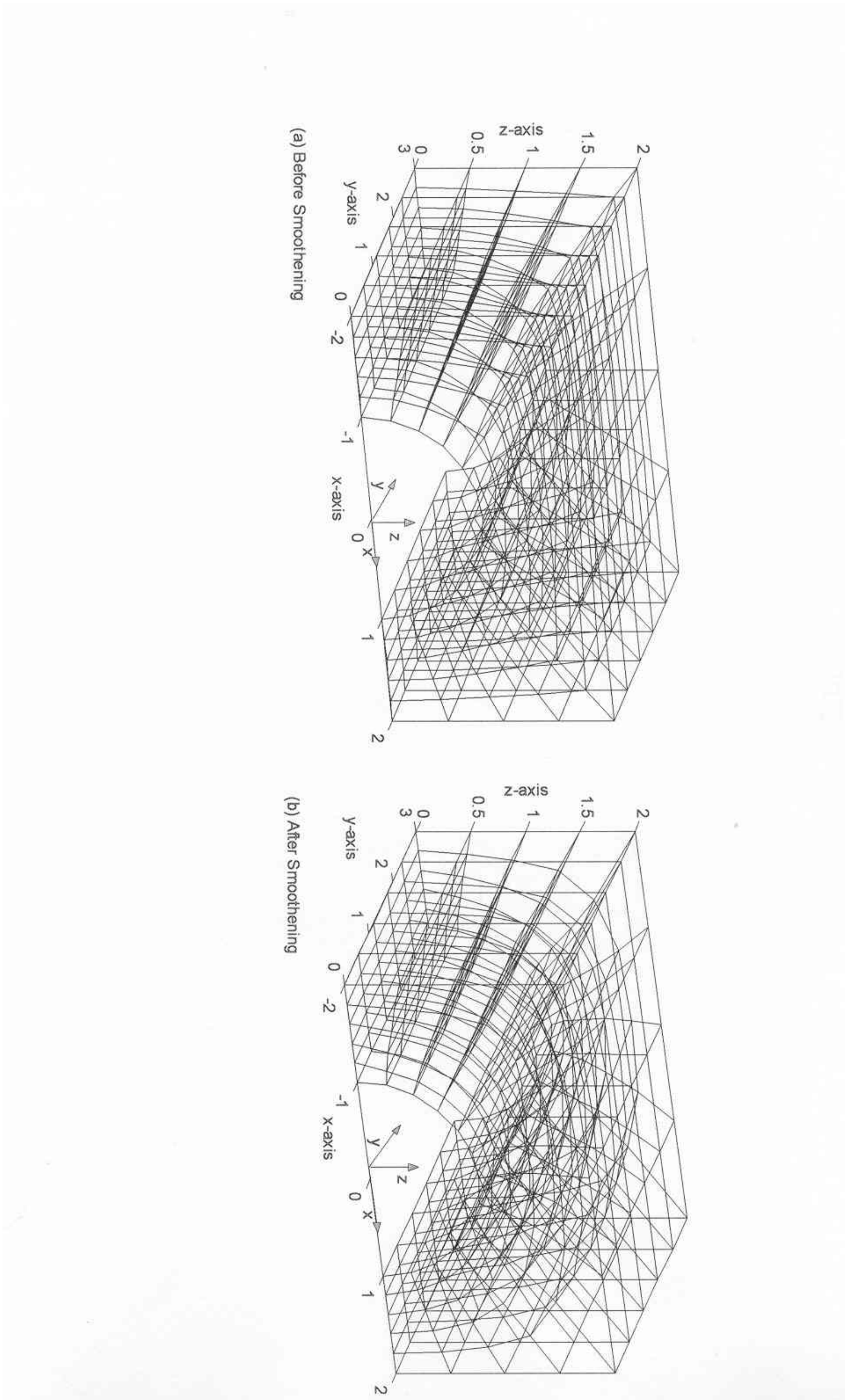
z-axis
y-axis
x-axis
(a) Before Smoothening
z-axis
y-axis
x-axis
(b) After Smoothening

Figure 8

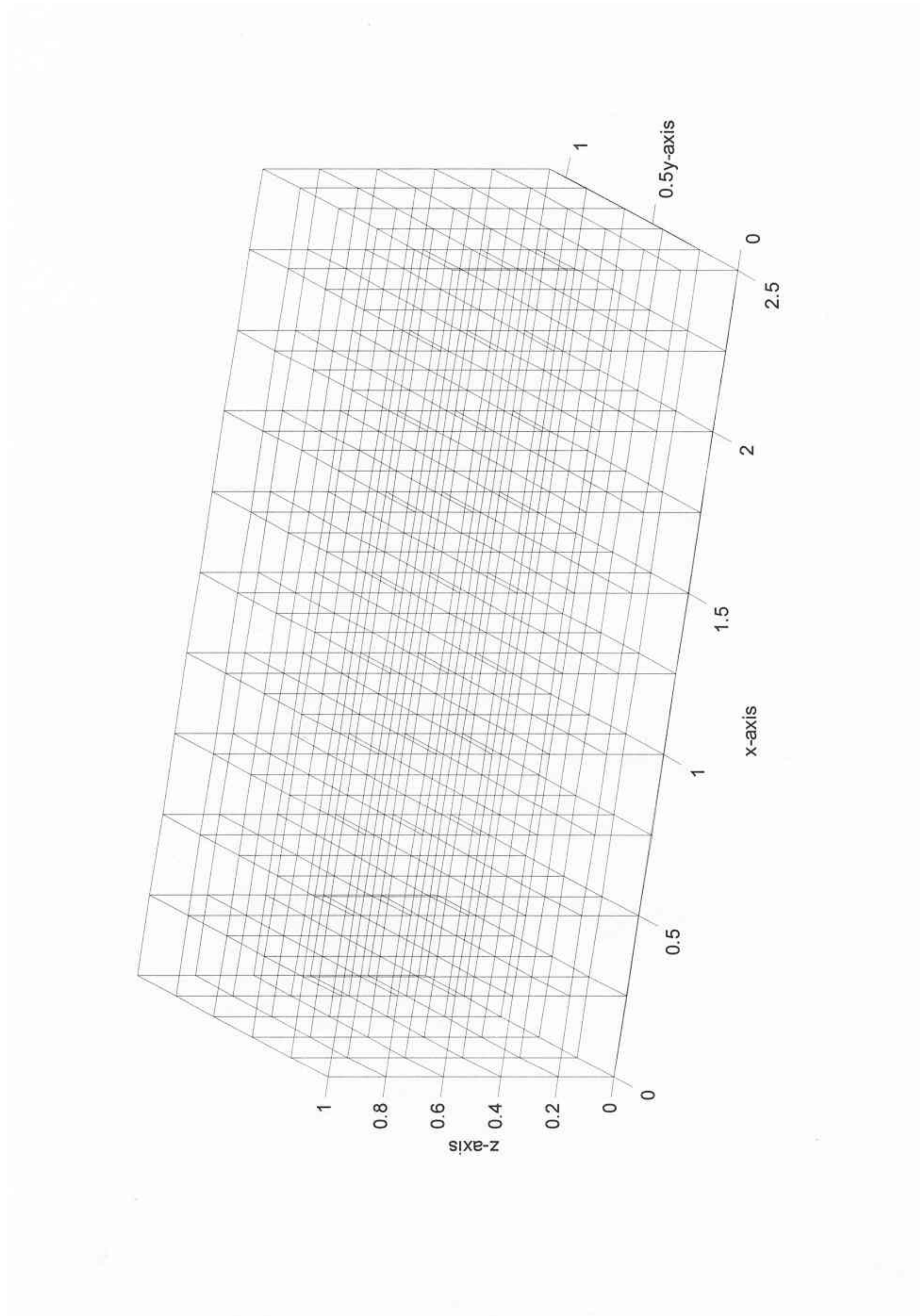

Figure 9

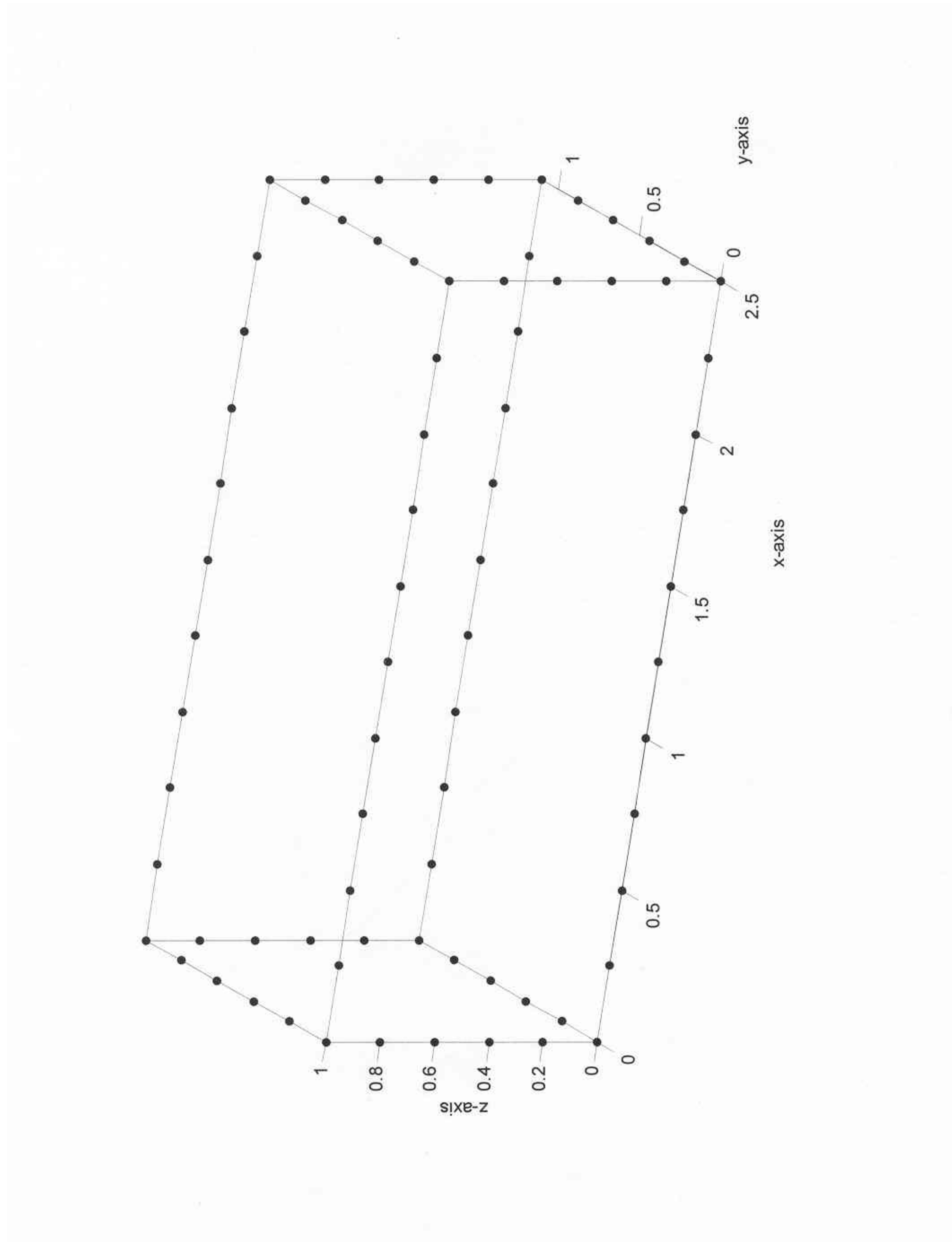

Figure 10

From: John Woodwark [jrw@johnwoodwark.com]
Sent: Τρίτη, 9 Σεπτεμβρίου 2003 2:45 μμ
To: C. G. Provatidis
Subject: Your CAD submission

Prof Provatidis,

Thank you for your manuscript "On the integration between CAD and CAE in engineering design". I have looked through the paper, and I have to say that I don't really think it's suitable for CAD journal.

Firstly, I find the title undescriptive. The "integration of CAD and CAE" might involve machining, DfA, STEP, customization... and 1001 other things: certainly not just FE. A more accurate title might be something like "A finite element based on Coons interpolation", and I think it then starts to become obvious why this is not really a CAD journal paper: certainly we carry *some* stuff about FE and meshing but, as you know, there is a large specialist literature. It's not an area in which we are looking to get more deeply involved -- and, I would say, especially not in new types of element.

Secondly, there seem to be some aspects of a review paper in this manuscript (not least its length), including a summary of your own work. I think your letter and the ms. are admirably straightforward about this, but it's not very clear that there remains enough new material to merit a journal publication. You do itemize the new contributions on p. 5, but I have to say that I consider these to be too specific to support the superstructure you have erected around them.

Thirdly, while the results in Tables 1 and 2 look impressive, I'm not entirely convinced -- even as a non-expert on FE -- by the comparisons. For instance I would certainly have liked to see computation times in the tables. And the components you are meshing certainly don't look particulaly challenging. When you are up against a very well established technology like 'conventional FE', I think you have to present a very thorough argument. But in any case I fear that CAD journal isn't the right place to do it.

I'm sorry to disappoint you.

--

Regards

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Athens, 12th September 2003

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Re: **CAD-FEA integration using Coons interpolation**,
by C.G.Provatidis (paper submitted to *Engineering with Computers*)

Dear Professor Shephard,

Attached please you find a paper submitted to *Engineering with Computers*. It proposes Coons interpolation as the "connecting tissue" between CAD (solid modeling) and analysis (FEA).

Perhaps this paper is long but it was extremely difficult for me to make it shorter.

1) First, the paper summarizes 15 papers of mine, where "Coons interpolation formula" was applied to build large 3-D boundary elements and 2-D macroelements (similar to BEM) in many field such as elastodynamics, acoustics and potential problems. Unfortunately, most papers have been published in Conference Proceedings (difficult to be retrieved) so that I felt the need to communicate them through the first part of this paper.

2) Second, the paper includes three novel features. The first feature concerns the use of Coons interpolation for *smooth* 2-D meshes. The second feature concerns the use of Coons interpolation for *smooth* 3-D meshes. The third feature concerns the development of large 3-D macroelements. Finally, large BEM and 3-D macroelements are compared with conventional BEM and conventional FEM solutions.

Finally, I think that Section 7 and Section 8 could be two independent papers, but I feel that the character of "Engineering with Computers" is not to deeply describe new elements but overall methodologies.

I confirm that this paper is original and it has not been submitted or published elsewhere.

Hoping that you will find the paper interesting for consideration in your Journal.

Sincerely

C. Provatidis   Encl: Paper in dublicate.

From: Marge Verville [vervim@rpi.edu]
Sent: Πέμπτη, 25 Σεπτεμβρίου 2003 8:46 μμ
To: Christopher Provatidis
Subject: EWC Submission #03-019

Dear Prof. Provatidis:

This will acknowledge receipt of your submission entitled:

"CAD-FEA integration using Coons interpolation"

for consideration for publication in Engineering with Computers.

Your manuscript has been assigned log number EWC03-019. Please use this number in any correspondence relating to the paper.

The review process has begun, and we will notify you of the results when they are finalized.

Thank you for your interest in Engineering with Computers.

Sincerely,

Mark S. Shephard
Editor

****
Sent by:

Marge Verville
Administrative Assistant
Scientific Computation Research Center (SCOREC) CII 7013 Rensselaer Polytechnic Institute
Troy, NY 12180-3590
518-276-6795 (voice); 518-276-4886 (fax) office@scorec.rpi.edu or vervim@rpi.edu

From: SCOREC Office [office@scorec.rpi.edu]
Sent: Πέμπτη, 28 Ιουλίου 2005 5:26 μμ
To: Christopher Provatidis
Cc: Mark Shephard
Subject: [Fwd: [Fwd: Log number EWC03-19: Reviewing Status]]
Attachments: EWC03-019 review.pdf; EWC03-019 reject.pdf; signature.asc

Dear Professor Provatidis:

Thank you for your inquiry. However, based on the results of the review process, your paper was rejected for publication in Engineering with Computers and the notice was sent to you by mail in October, 2004.
Electronic copies of the letter and the review are attached to this email.

Sincerely,

Marge Verville

-------- Original Message --------
Subject: Log number EWC03-19: Reviewing Status
Date: Wed, 27 Jul 2005 17:47:54 +0300
From: C.G. Provatidis <cprovat@central.ntua.gr>
To: <shephard@scorec.rpi.edu>

Re: Log number EWC03-19: CAD-FEA integration using Coons interpolation, by C.G.Provatidis (paper submitted to Engineering with Computers)

Dear Professor Shephard,

I am kindly asking you about the reviewing process of the abovementioned paper, which was submitted on 12th September 2003.

In the meanwhile, I have published (or "in-press") on the same subject, the following six additional papers which should be added in References:
1) C. Provatidis, Coons-patch macroelements in two-dimensional eigenvalue and scalar wave propagation problems, Computers & Structures, 82, 2004, 383-395.
2) C. Provatidis, Solution of two-dimensional Poisson problems in quadrilateral domains using transfinite Coons interpolation, Communications in Numerical Methods in Engineering, Vol. 20 (7), July 2004, pp. 521-533.
3) C. G. Provatidis, Three-dimensional Coons macroelements in Laplace and acoustic problems, Computers and Structures, Vol. 83 (2005) 1572-1583.
4) C. G. Provatidis, Analysis of box-like structures using 3-D Coons' interpolation, Communications in Numerical Methods in Engineering, (ON-LINE: April 2005).
5) C. G. Provatidis, Coons-patch macroelements in two-dimensional parabolic problems, Applied Mathematical Modelling (in-press, PDF is attached: article.pdf).
6) C. G. Provatidis, Three-dimensional Coons' macroelements: Application to eigenvalue and scalar wave propagation problems, International Journal for Numerical Methods in Engineering, (accepted).

Probably, these articles should be forwarded to the reviewers.

Apart from the reviewing aspects, a significant part of EWC03-019 includes original findings (such as several formulations of Coons interpolation, 3-D smoothening, BEM macroelements), on which further progress has been made.
Therefore, I have a dilemma how to proceed with my next paper submissions in other journals, i.e. should I refer to EWC03-019 as "submitted" or I have no chance for that?

As I will be on holidays between 1st and 15th August 2005, there is no need to answer immediately.

Looking forward to hearing from you soon.

Sincerely,

Christopher Gabriel PROVATIDIS

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Engineering with
Computers
An International Journal for Simulation-Based Engineering

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October 15, 2004

Prof. Christopher Provatidis
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9, Heroon Polytechniou Str., Zografos Campus
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Greece

RE: Paper EWC03-019

Title: "CAD-FEA integration using Coons interpolation"

Author(s): Christopher G. Provatidis

Dear Prof. Provatidis:

Based on the review, it has been recommended that your paper be rejected for publication in Engineering with Computers. A copy of the review is enclosed.

Thank you for your interest in Engineering with Computers.

Sincerely,

Mark Shephard

Mark S. Shephard
Editor

Springer
London Berlin Heidelberg New York Paris Tokyo Hong Kong Barcelona Budapest Milan Singapore
http://link.springer.de http://link.springer.co.uk

Review of EWC03-019:

Review comments for Engineering with Computers, “CAD-FEA Integration Using Coons Interpolation” by Provatidis

I recommend that the paper needs major revision and re-review.

The paper seems to wander between too many topics under the overly broad justification of integration of CAD and CAE. First is the review of bi- and tri-variate interpolation, followed by some node smoothing (“smoothening” in the paper), and then Coons macroelements. Within these sections are changing topics that cause confusion, and left me unsure of any real objective of the paper.

Also, it was unclear at times what is really new. The paper begins with a review of basic and well-known Coons interpolation. It then states that there are two novel features of the paper, namely generating smooth meshes within boxlike structures, and avoiding mesh generation by working in conjunction with only the twelve edges of the boxlike structure. In my opinion, this material does not seem new. It is at best a restatement or small improvement over known methods. The smoothing is simply nearest neighbor point averaging in XYZ space after the nodes are generated in equi-spaced parametric space, which as everyone knows creates uneven XYZ spacing due to the nonlinear mapping.

The novelty of the trivariate discussion concerned me because it has been my understanding that commercial finite element programs long ago used trivariate parameter elements (e.g. PATRAN?). So, it seemed that this too is not new.

The macroelement may be new and a noteworthy focus of the revised paper.

Finally, there is the problem of how current CAD solid models with their complex trimmed NURB surfaces can be “meshed” into sets of these boxlike structures. This is a complex and unsolved problem that may negate the proposed elegance of the trivariate boxlike element. This is not addressed in the paper – so it is not correct to claim CAD-CAE integration.