DETERMINATION OF EIGENFREQUENCIES IN THREE-DIMENSIONAL ACOUSTIC CAVITIES USING COONS-PATCH BOUNDARY SUPERELEMENTS

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1. SUMMARY

In its usual formulation, three-dimensional DR/BEM analysis requires the discretization of the boundary using constant, linear or quadratic elements. In this paper it is shown that transfinite Coons interpolation allows for the construction of large shape functions affecting a whole smooth boundary patch of quadrilateral shape. The nodal points are arranged along the patch boundaries only, so that the number of nodal points necessary to perform an analysis is drastically reduced. The efficiency of the proposed boundary superelements to deal with the eigenvalue acoustic problem is investigated. Test cases on a cubic and a rectangular cavity shape sustain the theory, where the proposed formulation is successfully compared with finite elements and exact analytical solution.

2. INTRODUCTION

Considerable progress has been made in recent years in developing the finite element (FEM) and finite difference methods (FDM) for acoustic cavity analysis [1]. However, since three-dimensional problems are characterized by the formidable demand on data preparation effort when FEM or FDM is applied, the Boundary Element Method (BEM) is perhaps the most powerful tool, as it demands only the discretization of the boundary of the cavity. Nevertheless, frequency-domain BEM analysis requires frequency-dependent kernels, a fact that leads to full matrices, which have to be re-calculated for each discrete frequency (nonalgebraic problem).

However, since 1982 this problem was overcome through a global set of conical radial basis functions (RBFs) that approximate the inertial term [2]. Furthermore, using a dual set it becomes possible to derive boundary-type mass matrices, so as BEM deals now with an algebraic problem. This technique was called “dual reciprocity boundary element method” (DR/BEM) [3] and was further extended apart from conical to spline and multiquadratic types [4]. So far, the radial basis functions were proven to be effective tools in multivariate surface interpolation and have wide applications in neural networks, computer graphics design, geoscience etc. [5]. A review paper on the evaluation of 29 two-dimensional interpolation methods was reported by Franke [6]. Furthermore, Micchelli [7] cleared up the issue of invertability of the resulting matrix using RBF interpolation, which gave a firm mathematical foundation to the development of the RBFs.

Usually, BEM requires a discretization of the boundary using constant, linear or quadratic elements. This paper is an attempt to further reduce the number of degrees of freedom involved in a BEM model in order to (i) reduce computer cost, (ii) increase communication-reliability between solid model (CAD) and analysis (CAE) and (iii) obtain a better control on shape optimization or inverse problems solution. Clearly, it will be shown that the use of bivariate Coons interpolation allows for the global interpolation of geometry, acoustic pressure and particle velocity within a whole quadrilateral patch (at the boundary of the cavity), so that the nodal points can be arranged along their boundaries only. For any interpolation assumed along each side of a certain patch, for example piecewise linear or B-splines, a global cardinal shape function can be estimated using simple analytical formulas [8,9]. So far, the method has been successfully applied to the solution of sound radiation (Helmholtz) problem [10]. It is the purpose of this paper to investigate the accuracy of this method in the eigenvalue extraction of 3-D acoustic cavities, despite past reservations for the 2-D problem [11,12].

The proposed method will be applied to cubic and rectangular cavities and compared with conventional finite elements.

3. BIVARIATE COONS INTERPOLATION – ISOPARAMETRIC MACROELEMENTS

The origin of Coons patch interpolation is middle 1960s, where it was applied to approximate a smooth curvilinear surface [13]. Since then, it was also applied to mesh generation tasks [14]. Here, it is assumed that the exact solution (acoustic pressure and particle velocity) over a boundary patch is smooth. Furthermore, the aforementioned interpolation is extended from geometry to the interpolation of the unknown variable. In other words, Coons interpolation formula offers the mathematical background to develop large isoparametric elements of arbitrary nodes (also small ones [8]).

Within a smooth curvilinear patch it is possible to establish a reasonable relation between the four surrounding boundaries and the internal points. Let us now consider a curvilinear *Coons*' patch that consists of four boundaries AB, BC, CD and DA, along the normalized *r*- and *s*-axes, with $0 \le r, s \le 1$, as shown in Figure 1. In this case, the coordinates $\mathbf{x}=(x,y,z)$ of the internal points can be interpolated in terms of the boundary and the two internal lines as follows [14,p.361]:

$$\begin{aligned}\mathbf{x}(r,s) &= (1-r)\,\mathbf{x}(0,s) + r\,\mathbf{x}(1,s) + (1-s)\,\mathbf{x}(r,0) + s\,\mathbf{x}(r,1) \\ &\quad - (1-r)(1-s)\,\mathbf{x}(0,0) - r(1-s)\,\mathbf{x}(1,0) - (1-r)s\,\mathbf{x}(0,1) - rs\,\mathbf{x}(1,1)\end{aligned} \tag{1}$$

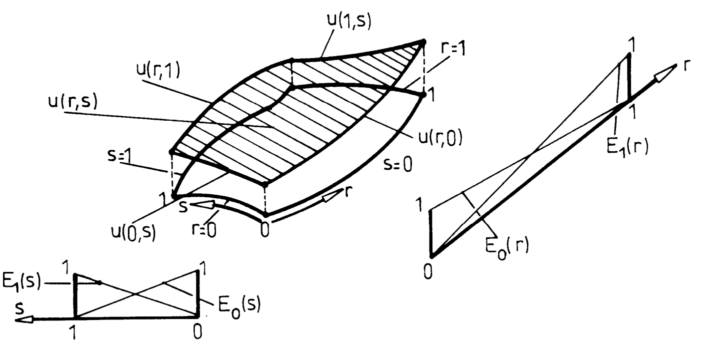

Figure 1: Boundary curves $u(0,s)$, $u(1,s)$, $u(r,0)$, $u(r,1)$ and linear 'blending functions' E_0 and E_1 of the macro-element.

In this work, the interpolation according to equation (1) is also valid to acoustic pressure as well as to particle velocity within any patch.

4. DR/BEM FORMULATION

The 3D wave propagation problem is described by the equation

$$(1/c^2)\partial^2 p/\partial t^2 - \nabla^2 p = 0 \tag{2}$$

and the boundary conditions

$$p = \overline{p}(t) \text{ on } \Gamma_1 \quad \text{and} \quad v = \overline{v}(t) \text{ on } \Gamma_2 \tag{3}$$

In (2), (3) p denotes the acoustic pressure, c the velocity of the wave propagation, ∇ the Nabla operator, t the time, and Γ_1 and Γ_2 parts of the boundary Γ, where the acoustic pressure respectively the velocity is prescribed .

Using the Galerkin's method, the partial differential equation (2) is transformed into

$$\int_V p_i^* \cdot \left[\left(1/c^2\right)\partial^2 p/\partial t^2 - \nabla^2 p\right] dV = 0,\ i = 1,2,..., \tag{4}$$

with p_i^* being the fundamental solution of 3-D Laplace equation, fulfilling the equation

$$\nabla^2 p_i^* + \Delta^i = 0 \tag{5}$$

where Δ^i is the Dirac function at the source point *'i'* of the domain. For the 3D case of this paper, p_i^* is given by

$$p_i^*(\boldsymbol{\xi}, \mathbf{x}_i) = 1/(4\pi r) \tag{6}$$

with $\boldsymbol{\xi}$ and $\mathbf{x}_i$ being the field and source points, respectively, and $r = |\boldsymbol{\xi} - \mathbf{x}_i|$ the distance between the above points.

Substitution of (5) in (4) in conjunction with Green's theorem leads finally to the following equation:

$$\int_V p_i^* \left(1/c^2\, \partial^2 p/\partial t^2\right) dV + c_i p_i + \oiint_\Gamma p \cdot \partial p_i^*/\partial n = \oiint_\Gamma p_i^* \cdot \partial p/\partial n\, d\Gamma \tag{7}$$

where c_i equals to ½, 1 or 0 for a (smooth) boundary, an internal or an external point, respectively.

In the conventional BEM, the boundary is discretized into a number of boundary elements where constant, linear or quadratic interpolation is assumed for both acoustic pressure and velocity. It is reminded that the normal component of the velocity, v_n, is proportional to the normal derivative of the acoustic pressure ($\partial p/\partial n = -j\omega\rho_0 v_n$), with ω denoting the angular frequency, ρ_0 the mass density of the still fluid and $j^2 = -1$.

Now, an interpolation should be made for the inertial terms through a series expansion

$$p(x,t) = \sum_{j=1}^{n} f(x, \xi_j)\alpha_j(t) = \sum_{j=1}^{n} f_j(x)\alpha_j(t) \tag{8}$$

where $\mathbf{x}$ is a point inside the cavity, ξ_j is a boundary node and $F(x, \xi_j)$ is a basis functions carried by the *j*-th boundary node. Note that numerical experience from 2D analysis has shown that results may improve when extending the sum in (8) through some few internal nodes [11,12,15,16].

Among the several alternative possibilities, this paper follows Nardini [2] as well as Banerjee et al. [17] and assumes that

$$f_j = f(x, \xi_j) = C - r \tag{9}$$

where C is the largest distance between any two points in the body and r is the distance between x and ξ_j.

Then, if the smooth boundary is discretized using n elements, in matrix form eq.(7) becomes

$$\hat{\mathbf{M}} \cdot \ddot{\boldsymbol{\alpha}} + \mathbf{H} \cdot \mathbf{p} = \mathbf{G} \cdot \mathbf{v} \tag{10}$$

where $\mathbf{H}$ and $\mathbf{G}$ are the conventional static matrices [19], while the elements of $\hat{\mathbf{M}}$ ($n\times n$) are given by

$$\hat{m}_{ij} = \left(1/c^2\right) \cdot \int_V p_i^* \cdot f_j\, dV,\ i, j = 1,2..., \tag{11}$$

Now, we define a dual functional set, ψ_j, to that of eq.(9), so that

$$\nabla^2 \psi_j = f_j \tag{12}$$

By substituting (12) into (11) and then applying Green's theorem, one finally obtains

$$\hat{m}_{ij} = \left(1/c^2\right)\cdot\left\{\oiint_\Gamma p_i^* \,\partial\psi_j/\partial n \,\mathrm{d}\Gamma - \left[c_i \psi_j(x_i) + \oiint_\Gamma \partial p_i^*/\partial n \cdot \psi_j\right]\right\} \tag{13}$$

So, according to (13), the elements of the mass matrix can be calculated as boundary integrals. Furthermore, if the continuous function ψ_j is approximated along each boundary element, in the same manner as the pressure and the velocity, eq.(12) is simplified to

$$\hat{\mathbf{M}} = \left(1/c^2\right)\cdot\left(\mathbf{G}\cdot\boldsymbol{\eta} - \mathbf{H}\cdot\boldsymbol{\psi}\right) \tag{14}$$

For the particular case of eq.(9), the dual functional set may be given as

$$\psi_j = r^3/12 - C\,r^2/6 \tag{15}$$

Remark: The above formulation is consistent to the general procedure proposed by Nardini and Brebbia [2] while an alternative approach through partial solutions proposed by Banerjee et al [17] is essentially identical, as has been shown by Polyzos et al. [18].

5. NUMERICAL IMPLEMENTATION

For the purposes of the numerical integration only, the patch is automatically divided into $N_r \times N_s$ cells where a second set of normalized co-ordinates $(-1 \le r', s' \le 1)$ is introduced [20]. So, the term $d\Gamma = |G(r,s)|\,dr\,ds$ in (2), with G denoting the Jacobian from cartesian to natural coordinates, is replaced by $|G(r,s)\cdot G'(r',s')|\,dr'\,ds'$, which requires a trivial (e.g., 2×2, 3×3, 4×4) Gaussian quadrature. Of course, special attention should be paid to identify the areas around each source where singularities occur.

6. EXAMPLES

Example 1: Cube of unit dimensions

The twelve edges of the cube were uniformly discretized using progressively 2, 4 and 6 segments. The corresponding number of nodal points is 20, 44 and 68. The constant '*C*' in (15) was taken equal to $\sqrt{3}$, which is the maximum distance between two points in the cube. Convergence quality is shown in Table 1.

Table 1: Calculated eigen-wavenumbers for a unit cube under free-free boundary conditions

Boundary Macroelement			FEM (343 nodes)	Exact Eigenwavenumber ($k = \omega/c$)
20 nodes	44 nodes	68 nodes		
0.	0.	0.	0.	0.
3.39	3.30	3.29	3.18	3.14
4.90	4.71	4.68	4.49	4.44
6.37	5.77	5.69	5.50	5.44

It can be noticed that the proposed macroelement converges towards the exact solution but it is less accurate than the FEM solution that corresponds to the case of 68 nodes. This difference appears also in case of conventional boundary elements and the reasons have been discussed in the past [11,12]. It is anticipated that when using higher order radial functions, the difference will decrease. In any case, the engineering purposes the results of the proposed method are acceptable.

Example 2: Rectangular cavity

A rectangular cavity of dimensions $L_x \times L_y \times L_z$ =2.5×1.1×1.0 was divided in 76 nodes using 10, 5 and 5 uniform segments along the edges parallel to *x*-, *y*- and *z*-axis, respectively. The calculated natural wavenumbers ($k = \omega/c$) and corresponding modes are shown in Table 2.

Table 2: Calculated eigen-wavenumbers for a rectangular cavity of dimensions 2.5×1.1×1.0 under free-free boundary conditions

Exact Eigen-wavenumbers	Mode	Boundary Macroelement (76 nodes)	FEM (396 nodes, 250 elements)
$\omega_1 = 0.000$	(0,0,0)	0.000	0.000
$\omega_2 = 1.2566$	(1,0,0)	1.2564	1.2618
$\omega_3 = 2.5133$	(2,0,0)	2.6258	2.5548
$\omega_4 = 2.8560$	(0,1,0)	3.0387	2.9032
$\omega_5 = 3.1202$	(1,1,0)	3.2549	3.1655
$\omega_6 = 3.1416$	(0,0,1)	3.3588	3.1935
$\omega_7 = 3.3836$	(1,0,1)	3.5554	3.4337
$\omega_8 = 3.7699$	(3,0,0)	4.0801	3.8672

Despite the fact that the proposed macroelement is less accurate than the FEM, for the same number of 76 nodes along the twelve edges, however for engineering purposes it is acceptable.

7. CONCLUSIONS

It was shown that global shape functions based on Coons interpolation over large patches could successfully substitute conventional boundary elements in the solution of eigenvalue extraction in acoustic cavities. The advantage of the proposed method is that it integrates the geometrical model with computational analysis, using degrees of freedom along those geometrical entities absolutely necessary to determine the shape of the cavity. So, instead of surface discretization, in many cases it is only necessary to deal with only a few lines (edges) and -in this sense- the dimensionality reduces from 3D (volume) to 1D (lines). It is anticipated that the proposed approach will be useful in practical cases of shape optimization, where a minimum number of variables will participate in nonlinear mathematical programming or other techniques.

8. REFERENCES

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