FREE-VIBRATION ANALYSIS OF THREE-DIMENSIONAL SOLIDS USING COONS-PATCH BOUNDARY SUPERELEMENTS

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1. SUMMARY

A new modified BEM technique is proposed for the extraction of eigenvalues in solid structures. Instead of dividing the whole boundary into elements of small size, it is proposed to decompose that into large curvilinear Coons patches. Nodal points and associated degrees of freedom are located along the boundaries of the aforementioned patches. In this way, the model size is drastically reduced for the two static matrices as well as the Nardini-Brebbia mass matrix. Moreover, investigation is performed on the quality of the mass matrix, that is, on its capability of preserving the real mass of the structure. The methodology is sustained by numerical examples on a rectangular structure where the well-known conical radial basis functions are used.

2. INTRODUCTION

It is well known that geometric modeling (CAD) and computer-aided analysis (CAE) are individually powerful, but they do not always work well together. In this context, large-scale interpolation, which operates directly on the geometric modeling representation, is welcome. With respect to the BEM, Casale and Bobrow [1] propose a division of the solid's boundary into a certain number of large patches and application of global interpolation over each of them. So, instead of the usual linear and quadratic boundary elements, they globally apply Lagrange polynomials; the so produced boundary elements have been called "*trimmed-patch boundary elements*". As numerical results have not been presented in their paper, it is hypothesized that Lagrange polynomials will probably have shortcomings due to their well-known oscillating behavior.

In this paper, instead of *global interpolation per patch* using Lagrange polynomials, a different interpolation is proposed for the BEM solution of eigenvalue elasticity problems. The origin of our method is based on CAD surfaces used in automotive industry, where blending function methods based on the ideas put forward by Coons [2] have been proposed to produce the so-called *transfinite elements* [3,4]. These large elements constitute an *extension* of the *isoparametric* elements and especially those ones called 'serendipity'. In other words, the well-known linear and quadratic boundary elements are simple members of the class of the proposed Coons-patch boundary macroelements.

The advantage of the proposed method is that the number of nodal points is dramatically reduced. For example, in case of six-sided solids, the degrees of freedom are arranged only along the twelve edges! So far, these elements were recently applied to elastostatics by the first author [5] but in this paper the method is extended to the determination of the eigenfreqiencies of a solid using the Nardini-Brebbia Dual Reciprocity Method (DR/BEM) [6,7].

The method was successfully applied to a free-free rectangular solid and the macroelement solution is compared with conventional nine-node Lagrange isoparametric boundary elements (using the same set of conical base functions), as well as conventional finite elements.

3. GENERAL DYNAMIC BEM FORMULATION

According the the Nardini-Brebbia methodology [6], the stress equilibrium equation

$$\partial\sigma_{ij} \big/ \partial x_j + b_i = \rho \ddot{u}_i \tag{1}$$

is tranformed into an integral formulation as follows:

$$\int_V u^*_{lk}\, \rho \ddot{u}_i \,\mathrm{d}V + c^i u^i_l + \oiint_\Gamma p^*_{lk}\, u_k \,\mathrm{d}\Gamma = \oiint_\Gamma u^*_{lk}\, p_k \,\mathrm{d}\Gamma + \int_V u^*_{lk}\, b_k \,\mathrm{d}V \tag{2}$$

Usually the term including the body forces vanishes, but if not, it can be easily transformed into the boundary using a proper Galerkin-vector or a similar technique. Moreover, the first integral appearing in (2) is a volume one and it can be handled by considering a series expansion in time (t) and space (X) as follows

$$u_i(X,t) = \alpha^j_i(t)\, f^{\,j}(X) \quad \text{[sum over the idle index } j\text{=1,…,}m\text{]} \tag{3}$$

with $f^{\,j}(X)$ denoting the radial basis function (RBF). In its initial form, the basis function had been chosen as a conical function given by $f^{\,j}(X) = C - r$, with C denoting a suitable constant and r the Euclidian distance between any field point and the 'j'-th source point, usually on the boundary of the structure. Further research on 3D structures suggested that C should be taken as the largest distance between any two source points of the body [7, p.1743].

According to Nardini and Brebbia [6], when finding a displacement field ψ^j_{li} (associated with traction field η^j_{li}) with the corresponding stress tensor $\tau^j_{\lim}$ such that $\tau^j_{\lim,m} = \delta_{li}\, f^{\,j}$, equation (2) leads finally to

$$\hat{\mathbf{M}}\,\ddot{\boldsymbol{\alpha}} + \mathbf{H}\,\mathbf{u} = \mathbf{G}\,\mathbf{t}\,, \text{ where } \hat{\mathbf{M}} = \rho\left(\mathbf{H}\boldsymbol{\psi} - \mathbf{G}\boldsymbol{\eta}\right) \tag{4}$$

Using the relationship between nodal displacements $\mathbf{u}$ and coefficients $\boldsymbol{\alpha}$ through a matrix **F** ($\mathbf{u} = \mathbf{F}\,\boldsymbol{\alpha}$), and pre-multiplying both members of equation (4) by the inverse of matrix **G** (assuming a smooth boundary), one obtains

$$\overline{\mathbf{M}}\,\ddot{\mathbf{u}} + \overline{\mathbf{K}}\mathbf{u} = \mathbf{t}(t) \tag{5}$$

where

$$\overline{\mathbf{M}} = \mathbf{G}^{-1}\hat{\mathbf{M}}\mathbf{F}^{-1} \quad \text{and} \quad \overline{\mathbf{K}} = \mathbf{G}^{-1}\mathbf{H} \tag{6}$$

For the purpose of this paper, we use a suitable matrix **L**, well-known from the BEM/FEM coupling procedures (e.g., [8,p.274]), which achieves to transform the time-dependent tractions $\mathbf{t}(t)$ into boundary nodal forces $\mathbf{f}(t)$ as follows

$$\mathbf{f}(t) = \mathbf{L}\,\mathbf{t}(t) \tag{7}$$

By substituting equation (7) into equation (6) the latter becomes

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t) \quad \text{with} \quad \mathbf{M} = \mathbf{L}\overline{\mathbf{M}}, \mathbf{K} = \mathbf{L}\overline{\mathbf{K}} \tag{8}$$

So, the DR/BEM leads finally to a matrix formulation similar to that of the FEM.

4. COONS INTERPOLATION – NUMERICAL IMPLEMENTATION

The above formulation is valid for any interpolation involved along the boundary. One of the possibilities is to apply the Coons interpolation formula [2] that takes place in a curvilinear quadrilateral patch and approximates the coordinates of an internal point with those of its boundary through the equation [9,p.361; 10]:

$$\begin{aligned}\mathbf{x}(r,s) &= (1-r)\,\mathbf{x}(0,s) + r\,\mathbf{x}(1,s) + (1-s)\,\mathbf{x}(r,0) + s\,\mathbf{x}(r,1) \\ &\quad -(1-r)(1-s)\,\mathbf{x}(0,0) - r(1-s)\,\mathbf{x}(1,0) - (1-r)s\,\mathbf{x}(0,1) - rs\,\mathbf{x}(1,1)\end{aligned} \tag{9}$$

As also mentioned in [10] where more details may be found, in this paper equation (9) is extended from geometry to both displacements and tractions within the same patch. In the sequence, it is trivial to obtain global cardinal shape functions $\Phi_k(r,s)$, of which typical shapes may be found in [11] and elsewhere. So, the geometry $\mathbf{x}(x,y,z)$, the displacement vector $\mathbf{u}(x,y,z)$ and the traction vector $\mathbf{p}(x,y,z)$ inside a Coons patch, are all approximated by:

$$\mathbf{x}(r,s) = \sum_{k=1}^{K} \mathbf{\Phi}_k(r,s)\,\mathbf{x}_k, \mathbf{u}(r,s) = \sum_{k=1}^{K} \mathbf{\Phi}_k(r,s)\,\mathbf{u}_k, \mathbf{p}(r,s) = \sum_{k=1}^{K} \mathbf{\Phi}_k(r,s)\,\mathbf{p}_k \tag{10}$$

with $\Phi_k(r,s)$ denoting the shape functions within the whole patch, $\mathbf{u}_k$ nodal degrees of freedom appearing only at the boundaries of the patch, while r and s being its normalised $(0 \le r,s \le 1)$ curvilinear co-ordinates [11].

With respect to the numerical implementation, for the purposes of the numerical integration only, the patch is divided into $N_r \times N_s$ cells where a second set of normalized co-ordinates $(-1 \le r', s' \le 1)$ is introduced [5]. So, the term $d\Gamma = |G(r,s)|\,dr\,ds$ in (2), with G denoting the Jacobian from Cartesian to natural coordinates, is replaced by $|G(r,s) \cdot G'(r',s')|\,dr'\,ds'$, which requires a trivial (e.g., 2×2, 3×3, 4×4) Gaussian quadrature. Of course, special attention was paid to identify the areas around each source where singularities occur.

5. EXAMPLES

As an example, a rectangular structure of dimensions 3×3×2 under free-free boundary conditions was considered. Material properties were taken as follows: $E/\rho = 10^4$, $\nu = 0.30$. The Cartesian axes were chosen so that x and y are parallel to the edges of length three while z-axis is parallel to the edge of length two. The edges were uniformly divided into N_x, N_y and N_z segments, respectively. Wherever comparison is performed, the same number of nodal points exists along the twelve edges of the rectangular.

Table 1 presents the results for several discretizations (N_x, N_y and N_z) using both the proposed Coons BEM macroelements (constant in radial basis function: $C = \sqrt{3^2 + 3^2 + 2^2} = \sqrt{22}$) and finite elements with the same number of nodes along the edges. Obviously, due to the free-free boundary conditions, the first six eigenvalues equal to zero. Moreover, since the next exact eigenvalues are not known (contrary to the acoustic problem), relative errors were calculated with respect to a fine finite element mesh (N_x=15, N_y=15 and N_z=10, i.e. 2816nodes and 2250 elements).

One can notice in Table 1 that, for a small number of nodes, the proposed Coons-patch macroelement is more accurate than the FEM solution. As the number of nodes increases, the first nonzero eigenvalue (calculated using the proposed Coons-BEM formulation) is still adequately accurate but the higher ones become slightly less accurate than FEM solution, however being acceptable for engineering purposes. It is remarkable that the FEM solution overestimates the eigenvalues (monotonically converges from higher values) while the BEM solution does not.

Table 1: Calculated eigenvalues (ω^2) using Coons-patch macroelements and finite elements with the same number of nodes (N_x, N_y and N_z segments) along the twelve edges of the rectangular.

"EXACT" EIGEVALUES (FEM: 2816 nodes)	CALCULATED EIGENVALUES (ω^2) RELATIVE ERRORS IN (%)			
	(N_x=6, N_y=6 and N_z=4)		(N_x=9, N_y=9 and N_z=4)	
	Coons-patch BEM macroelement (60 nodes)	FEM (245 nodes, 144 elements)	Coons-patch BEM macroelement (84 nodes)	FEM (500 nodes, 324 elements)
2615.8	+1.30	+5.07	-0.93	+2.70
5101.7	-0.41	+7.28	-3.71	+3.01
6632.8	+1.83	+5.11	+2.46	+1.77
7284.4	-3.19	+7.69	-4.46	+3.77

Furthermore, for comparison purposes the same problem was solved using conventional boundary elements. In order to avoid shortcomings associated to the corners, discontinuous none-node Lagrangian elements were chosen. Each surface of the rectangular was uniformly divided into 2×2=4 boundary elements, leading to totally 24 elements (98 geometrical nodes and 216 collocation nodes). A sensitivity analysis was performed with respect to the position of the outer collocation points, which are defined in normalized coordinates within the interval $[\xi,\eta]=[-1,+1]$. The relative results are shown in Table 2.

Table 2: Influence of the position of collocation points in discontinuous nine-node Lagrangian boundary elements on the calculated eigenvalues. Radial basis functions were considered with the same constant C as in Table 1. Mesh consists of 24 boundary elements and 216 collocation nodes.

CALCULATED EIGENVALUES (ω^2)			
$\xi,\eta=\pm 0.50$	$\xi,\eta=\pm 1/\sqrt{3}$	$\xi,\eta=\pm 0.75$	$\xi,\eta=\pm 0.85$
2319.9	2315.7	2328.4	2346.1
4320.3	4318.7	4383.5	4448.1
5741.3	5583.4	5360.6	5261.2
5956.9	5863.5	5720.6	5652.5

It can be noticed in Table 2 that, for this mesh the conventional discontinuous boundary elements are not adequately accurate, as the error in first calculated eigenvalue is about 10 percent. This happens despite the fact that the participating collocation nodes are 216:60=3.6 times more than the coarse mesh in the proposed Coon-patch macroelement. It is also remarkable that when equation (8) was applied, the total mass was found to be equal to 35.2 times the mass density, which is less than the real mass [3 times the Volume = 3.(3×3×2) = 54.0 times the mass density!]. In other words, there is a considerable lack of mass (about one-third is missing), a finding being consistent to previous two-dimensional observations [12,p.123; 13]. Attempt was also made to determine the total mass for the simplest case of one boundary element per side but the matrix **F** could not be inverted.

In the sequence, we tried to increase the number of discontinuous conventional boundary elements from 24 to 42 (378 collocation-nodes, i.e. 1134 degrees of freedom). In this case, the computer effort became extremely high and the mass matrix could not be immediately inverted. Several multipliers were applied (e.g. 3×10^6) so that the determinant was estimated of the order of 10^{-70}, but nevertheless the quality of the results was not satisfactory because of many complex values found.

6. CONCLUSIONS

A new formulation for large boundary elements was presented and applied to the extraction of eigenvalues of elastic solid structures. It was proposed to divide the boundary of the structure into a small number of large patches at the boundaries of which the collocation points are arranged. In this way a significant reduction of the degrees of freedom was achieved. In case of simple rectangular-like structures, the dimensionality of the problem is drastically reduced from 3-D to 1-D, as only the twelve edges should be discretized. In other words, the proposed method achieves to deal with only the solid model and it therefore minimizes possible errors during data-transfer from the geometrical to the analytical model. The method was successfully applied to a rectangular solid structure and was found to be more accurate than conventional discontinuous boundary elements and finite elements. In the latter comparison, the same number of nodal points was considered along the twelve edges of the structure. After these encouraging results, the proposed method should be thoroughly tested for several support conditions and geometrical shapes. It should also be compared with other types of conventional boundary elements.

7. REFERENCES

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