Origins and Application of Geometry in the Thera Prehistoric Civilization Ca. 1650 BC

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Abstract. The present paper offers strong evidence that there was a particularly advanced, for the era, sense and application of geometry in the prehistoric civilization of the island of Thera (Santorini), Greece, ca. 1650 BC. First, by applying an original method, it is demonstrated that specific shapes, depicted on so far unpublished wall paintings initially decorating the third floor of Xeste 3, correspond to advanced geometric configurations with remarkable accuracy. Thus, it is shown that there are configurations corresponding to linear spiral prototypes, others matching elliptical prototypes and sets of points lying on isogonal lines that are radii of regular polygons with 48, 32, and 24 angles. Subsequently, it is shown that the use of geometric archetypes for drawing played a prominent role in the Late Bronze Age Thera civilization. In fact, it is demonstrated that celebrated wall paintings have border lines that impressively match a limited number of linear (Archimedes') spirals, hyperbolas, and ellipses in a piecewise manner. This practically excludes the probability that these wall paintings were drawn by freehand, while, on the contrary, it strongly suggests that they were mainly drawn by means of geometric stencils.

1. *Introduction*

In the south Aegean Sea, a celebrated Hellenic sea with more than 3,000 islands, the beautiful volcanic island of Thera (or Santorini) is found. In this island, a very important civilization grew in the prehistoric era, in the third and second millennium BC. Archaeological evidence indicates that in this civilization there was an impressive, for the era, accumulation of knowledge of pottery making, metallurgy, architecture, navigation, geography, astronomy, etc. (Doumas 1992). In prehistoric Thera, the arts also flourished, and were always, along with architecture and spatial organization, anthropocentric and essentially respected human scale.

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The growth of Theran civilization was abruptly and brutally stopped by the tremendous volcanic eruption, the greatest known in human history, which occurred around 1630 BC. The volcanic eruption was so fierce that considerable layers of Theran volcanic ash have been found at extreme distances from the epicenter, from Sudan to the eternal ice of Greenland. Consequently, the whole surface of the very island of Thera was completely covered by a thick volcanic ash layer up to 15–20 m in height. Thus, the prehistoric settlements and towns existing on the island were totally buried under a huge amount of volcanic ash, very much like Pompeii in south Italy.

A very important and very well-preserved prehistoric town was found by Professor S. Marinatos in Akrotiri, in 1967. Since then, continuous effort has brought to light a great number of archaeological finds of immense value. The most important of these finds seems to be the artistically superb paintings, which adorned the internal walls of the Akrotiri houses. These wall paintings are excavated in fragments, since the decorated walls have collapsed due to the strong earthquakes that preceded the volcanic eruption. These fragments are usually in excellent condition, since the thick layer of volcanic ash ensured stable conditions of temperature and humidity, creating an ideal conservation environment for both the plaster and the pigments. Thus, a high degree of recovery and restoration is nowadays possible but only via a painstaking and time-consuming process.

The assemblages of wall paintings from Akrotiri attest to the Theran artists' interest in a remarkable diversity of subjects: abstract patterns, inanimate objects and structures, plants, animals, human and mythological figures, geometric motifs, etc. There is a set of unpublished and not yet restored wall paintings belonging to Xeste 3 (House number 3), in which various geometrical figures are depicted such as spirals, circles, parallel lines, cycloids, crescent-like shapes. These wall paintings have been excavated highly fragmented, with several parts unfortunately missing. In fact, tenths of thousands of fragments are likely to belong to this set of wall paintings, while the overall image representation is not known with certainty at the moment. As a consequence, the restoration of this set of wall paintings is particularly difficult, so an original information system has been developed in order to assist the fragment-matching procedure (Papaodysseus *et al.* 2002).

Careful examination of the depicted geometrical shapes manifests that they were drawn with a very clear-cut and steady line, a noticeable order, and a noteworthy repeatability. This implied the idea that the artist (or artists) perhaps used geometrical methods and/or tools to draw parts of these wall paintings. In order to attest this conjecture, a novel methodology, as well as a class of original criteria, has been developed by the authors. It is shown below that application of this method offers strong evidence that

- All depicted spirals correspond, with impressive accuracy, to linear ones obeying Equation 1.
- There are configurations indicating knowledge of drawing isogonal (equiangular) lines and, in particular, those corresponding to a canonical 48-, 32-, and 24-gon.

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- The borders of most wall paintings initially decorating the internal walls of Xeste 3 were drawn with the help of geometrical prototypes corresponding to hyperbolas, ellipses, and linear spirals.

The details of the methodology from the mathematical and computer engineering point of view, as well as its archaeological implications, will be presented in another paper. In this paper, we proceed by setting the historical context from a mathematics point of view, in order to underline the essence of the findings reported in this paper. We would like to point out that, in the following discussion, for reasons of simplicity, we will refer to ‘the artist(s)’ as the hypothetical person(s) who had the necessary geometric sense and ingenuity to conceive, design, and implement the geometrical prototypes and draw the wall paintings in question. Of course, the person who made the geometrical designs may have been a completely different person than the one who actually drew the wall paintings or than the one who constructed the geometrical prototypes. It is evident that the interest of the present essay is in the advanced sense of geometry in this Late Bronze Age civilization, as well in the ingenious person(s) who conceived and applied this knowledge to drawing complicated figures on the wall.

2. *Related Geometry and Mathematics in the Prehistoric and Historic Era*

Simple geometrical prototypes, even in a primitive form, arose in various human activities even from the Paleolithic era, as a number of related finds manifests (Keller 2001). However, a noteworthy organization of geometric and arithmetic knowledge seems to have occurred in a far more recent era. Thus, it is well known (e.g. Neugebauer 1935–1937; Thureau-Dangin 1938; Neugebauer and Sachs 1945; Bruins and Rutten 1961) that in the old Babylonian (OB) period (ca. 2000–1600 BC), Babylonians dealt with a variety of problems of algebra and geometric algebra such as area computation, square root approximation, attempts to solve quadratic and logarithmic equations. Our knowledge of OB mathematics is based on numerous archaeological finds, such as the celebrated Plimpton 322, YBC 7289, AO 17264, and the considerable scientific effort made for their interpretation. Thus, Plimpton 322 was initially considered to provide a set of Pythagorean triads (Neugebauer and Sachs 1945); however, in Friberg (1981, p. 302) it is suggested that ‘the purpose of the author of Plimpton 322 was to write a “teachers aid” for setting up and solving problems involving right triangles’. Alternatively, Buck (1980, p. 344) suggests that ‘the Plimpton tablet has nothing to do with Pythagorean triplets or trigonometry but, instead, is a pedagogical tool intended to help a mathematics teacher of the period make up a large number of “igi-igibi” [i.e. reciprocal pair] quadratic equation exercises having known solutions and intermediate solution steps that are easily checked’. Moreover, in Robson (2001, p. 176) where a set of criteria for judging the interpretation of an archaeological find is introduced as well, it is stated that ‘If we believe that Plimton 322 was intended to be a list of parameters to aid the setting of school mathematics problems

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(and the typological evidence suggests that it was), the question "how was the tablet calculated?" does not have to have the same answer as the question "what problems does the tablet set?" The first can be answered most satisfactorily by reciprocal pairs, as first suggested half a century ago, and the second by some sort of right-triangle problems. That is perhaps as far as we can go on present evidence: without closer parallels we run the risk of crossing the fuzzy boundary from history to speculation. The mystery of the Cuneiform Tablet has not yet been fully solved' (Robson 2001, p. 202).

In AO 17264, Babylonians dealt with the problem of bisecting trapezia in order to solve practical problems like generating equal inheritance shares. In their attempt to do so, they dealt with the solution of a system of three second-order equations. As shown in Brack-Bernsen and Schmidt (1990), this problem is beyond the capability of Babylonian mathematicians, and it looks as if they have given up in despair their attempt at solving this problem and just given some meaningless computations that lead to a correct result.

Concerning YBC 7289, in Friberg (1982) it was first described that it was 'a lenticular school tablet with a geometric drawing displaying the very good approximations $\sqrt{2} \cong 1.24\,51\,10$ [and $1/\sqrt{2} \cong .42\,25\,35$]' (in sexagesimal system). In Fowler and Robson (1998, p. 369), it is pointed out that the function of YBC 7289 may be a 'rough work written by a student while solving a school problem'. Fowler and Robson using the Cuneiform Tablet BM 15285, an OB illustrated geometrical 'textbook' containing a number of problems on finding the areas of certain figures inscribed in squares, have reached the result that 'the author of YBC 7289 most probably took the value of $\sqrt{2}$ from a reference list'.

According to various researchers (Szabo 1968; Heath 1921/1981; Exarchakos 1997), starting from Proclus ('Πρόκλος'), Geometry in Egyptians was even in Thales' time confined to an area computing stage ('Γης Μέτρησις'); in other words, it was only an ensemble of empirical rules for computing surfaces, frequently with errors.

On the other hand, there are a number of researchers (Zeuthen 1896; Neugebauer 1936), who claim that one can interpret parts of Greek mathematics typified by book II of Euclid's 'Elements' as translations of Babylonian algebraic identities and procedures into geometric language. This position was fiercely supported by Unguru (1975), who argued that modern accounts of Greek mathematics have been so strongly affected by the concept of geometric algebra that it is now necessary to rewrite the whole subject. The Unguru point of view has suffered a number of serious attacks by Van der Waerden (1976a), Freudenthal (1977), and Weil (1978). Mueller (1981, p. 44) reached the conclusion that 'a strictly geometric reading of *The Elements* is ... sufficiently plausible to render the importation of algebraic ideas unnecessary'. Berggren's view in Berggren (1984, p. 398), where a very interesting and clear survey of Greek mathematic history is presented, is the following: 'to establish geometrized algebra as a historical fact still requires that considerable research be done on the time and method of transmission of Babylonian mathematical knowledge to the Greek world'.

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Next, a brief summary of geometric knowledge in Classical Ages related to the content of the current paper is presented. In the following, when a phrase of the type 'mathematician or philosopher X first stated and/or proved Proposition A' is used, we mean that it is historically well founded that statement or proof of proposition A is onomastically attributed to person X.

Thales (Θαλής) 624–546 BC seems to be the first who introduced the concept of proof in Geometry. He first stated that the circle is bisected by its diameter. One cannot exclude that this property may have been suggested to Thales by the appearance of certain figures of circles divided into a number of nearly equal sectors by two, four, or six diameters. Such figures were, for example, found on Egyptian monuments or represented on vessels brought by tributary monarchs in the 18th dynasty period (ca. 1560–1320 BC) (Heath 1921/1981 I, p. 131).

Oinopides of Chios in the fifth century BC is so far considered, even by Proclus, the first to draw the perpendicular to a given straight line from a point outside it using compasses. He is also considered to be the first who bisected a given angle.

Next, let us consider the following widely accepted proposition in modern mathematics: consider a circle and a canonical polygon, e.g. an equilateral triangle or a square inscribed in it. On each side AB of the inscribed triangle or square, consider the perpendicular from the circle center O and extend it until it meets the arc of the smaller segment of the circle subtended by the side AB, at a point M. In other words, construct an isosceles triangle MAB subtending the initial canonical polygon, with its vertex on the aforementioned smaller arc. In this way, one constructs a canonical polygon with double the number of sides of the initial one. Repeat the previous construction with the new canonical polygon, thus obtaining an inscribed canonical polygon with four times as many sides as the original polygon had and so forth.

Antiphon thought that in this way the area of the circle would be used up, and we should eventually have a polygon inscribed in the circle the sides of which would, owing to their smallness, coincide with the circumference of the circle. Archimedes, later on, in his celebrated work 'Κύκλου Μέτρησις' (Measurement of a Circle) used the aforementioned procedure in order to define and compute π toward his successful attempt to calculate the area of a circle and the length of its circumference. However, many authors consider Antiphon the father of the idea of exhausting an area by means of inscribed canonical polygons with an ever increasing number of sides, an idea upon which Eudoxus founded his method of exhaustion (Heath 1921/1981; Spandagos, Spandagou, and Travlou 1994/1997/2000).

Concerning spirals, it is well known that spiral shapes appear in various prehistoric civilizations, even centuries before 1650 BC, namely, before the prehistoric Thera civilization. However, we emphasize that the shape of the linear spiral does not exist in nature and that, to the best of our knowledge, the Thera wall paintings considered in this work constitute the first case where drawn figures approximate to such an impressive degree ideal geometrical shapes or curves obeying a certain corresponding equation. In Classical

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Age Geometry, the conception of the ‘torus spiral’ is attributed to Archytas the Tarantinus in the fourth century BC. In connection with the linear spiral, the one nowadays bearing Archimedes’ name, it seems that the idea of its construction and use is, so far, attributed to Konon (Κόνων) from Samos in the third century BC. In fact, Pappus (Πάππος) Alexandrinus, in his work ‘Συναγωγή’ (Collection) in the fourth century AC, states that the theorem about the plane spiral was proposed by Konon and proved by Archimedes. Indeed, in ‘On Spirals’ Archimedes defines the linear spiral and gives fundamental properties connecting the length of the radius vector with the angles through which it has revolved. He gives results on tangents to the spiral as well the area of portions of the spiral. Moreover, he uses this spiral for squaring the circle by rectifying a circle with the use of the spiral tangents (Heath 1921/1981; Band, Jones, and Bedient 1988; Spandagos, Spandagou, and Travlou 1994/1997/2000).

Concerning conics, it seems that the first who conceived them and realized that they result from the intersection of a cone with a plane was Menaichmos around 350 BC. Euclid in around 300 BC seems to be one of the first who wrote about them. In fact, according to Pappus (320 AD), ‘the four books of Euclid’s Conics, were completed by Apollonius, who added four more books of Conics’ (Heath 1921/1981). The names of the three conic types (ellipse, hyperbola, parabola) are attributed to Apollonius. He, moreover, stated and solved 10 problems concerning contacts. The solutions of these problems, even nowadays, require complicated methods.

3. *Advanced Knowledge and Application of Geometry in the Prehistoric Thera Civilization*

3.1 *Knowledge of Constructing an Archimedes (Linear) Spiral Ca. 1650 BC*

We start by noticing that the general equation of a spiral in polar coordinates is actually the one of a circle, where the radius is not constant, but, on the contrary, it is an increasing function of the polar angle θ. In certain cases, θ itself may be a function of other independent variables as well. Thus, clearly, from the theoretical point of view, there are infinitely many types of spiral.

Concerning the spirals depicted in Akrotiri, Thera wall paintings (see Figures 1a, b, and 2a, b), after thorough examination we noticed that they are very clear-cut, smooth, and stable-line figures. This, eventually, imposed the idea that the artist may have used a set of tools and/or geometrical methods to draw these spirals. In order to test this conjecture, we have chosen a set of prototype spirals, whose construction we felt that, even though requiring a considerable degree of novelty, was not completely prohibited by the means of the era. Thus, we have initially considered a potential prototype to be

(1) The spiral obtained by unwrapping a thread around a peg. Approximations of such a curve can be, clearly, encountered in nature.

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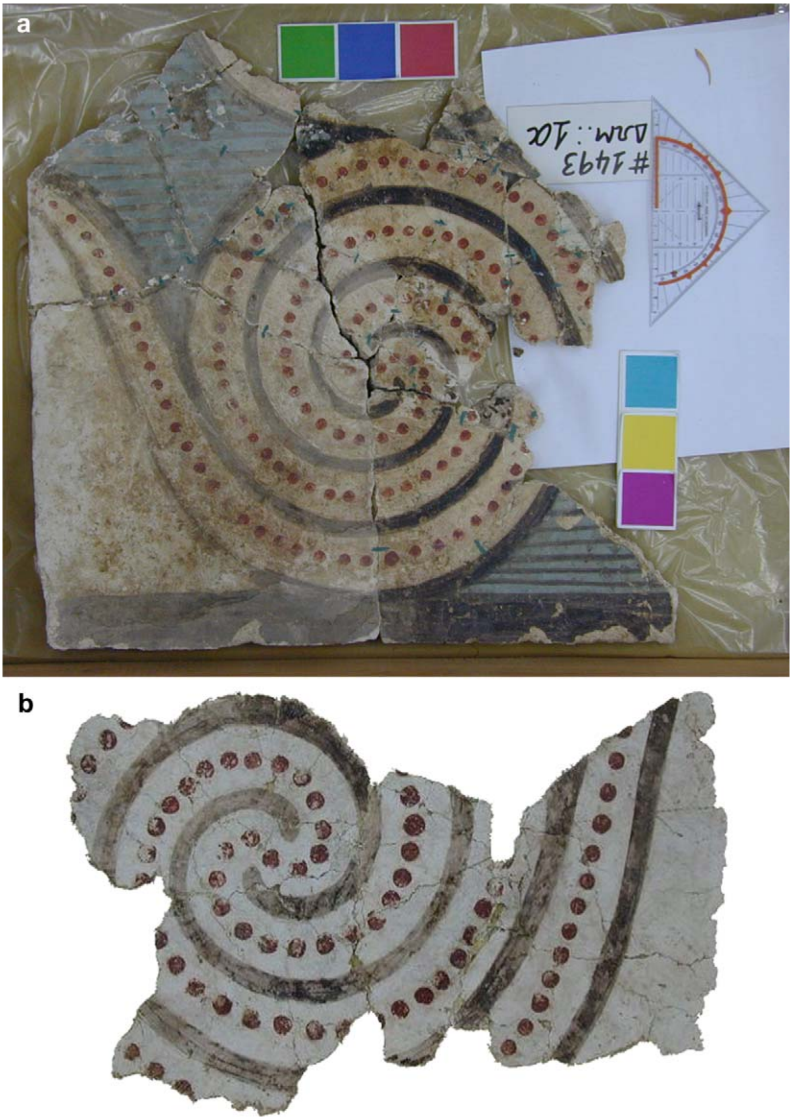

Fig. 1. (a) A LP1 class actually depicted on wall painting spirals, the S238. (b) Another LP1 class spiral, the S022.

(2) The logarithmic or exponential spiral, whose polar equation is such that the radius is an exponential function of the polar angle, θ. This spiral form is encountered in nature, too, e.g. in the seashells. The construction of such a curve requires a respectable amount of inspiration and novelty for the era.

(3) The linear or Archimedes spiral, whose polar equation is such that the radius is a linear function of the polar angle, θ, e.g.

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$$x(\theta) = x_0 + \kappa(\theta - \theta_0) * \cos(\theta - \phi_0)$$
$$y(\theta) = y_0 + \kappa(\theta - \theta_0) * \sin(\theta - \phi_0) \quad (1)$$

where κ is a constant, θ_0 is the angle corresponding to the starting point of the spiral part, and ϕ_0 is another angle that depends both on the starting point and the probable rotation angle. Such a spiral is not encountered in nature, and its conception and construction requires particular ingenuity.

In order to test if anyone of the above prototype spirals corresponds to the actually drawn ones, we have set and used three original criteria: two of statistical nature and one based on the maximal connected discrepancy of the prototype and the drawn spiral. To apply these criteria, the equation of each prototype spiral was first let converge in its parameter space to the position that best approximates the spiral part under consideration. The results obtained by the application of the aforementioned procedure and the related criteria can be summarized as follows:

(1) The hypothesis that the spirals depicted on the wall paintings were drawn by unwrapping a thread around a peg has been rejected for most available spiral parts.

(2) The hypothesis that the drawn spirals correspond to logarithmic prototypes has also been rejected for most of the available spiral parts. In any case, the approximation of the depicted spiral parts offered by the exponential prototype is clearly suboptimal to the approximation offered by the linear model. To set ideas, consider an actually depicted spiral, say AS, with a radius ranging from 2.6 to 20.4 cm and an average radius of 11.6 cm approximately, together with the exponential spiral, say ES, best matching it. If one uses the standard Euclidean norm in the plane, then the average distance between the drawn spiral AS and the exponential spiral ES is around 1.0–1.1 cm. The related average percentage error is about 11%, while the maximum percentage error is about 40%.

(3) The hypothesis that the depicted spiral parts correspond to a linear (Archimedes) prototype can be accepted for all available corresponding drawings. The approximation the linear prototypes offer to the depicted spiral parts is particularly good indeed, as shown in Figures 3–6. In fact, application of the aforementioned criteria and the related analysis shows that the available spiral parts correspond to two classes of prototype linear spirals bearing, of course, different sets of characteristics. We shall call these Linear Prototype 1 (LP1) and Linear Prototype 2 (LP2). In addition, we will call CLP1 the Class of spiral parts that best matches LP1 (Figures 1a, b) and CLP2 the class of spiral parts best approximated by LP2 (Figures 2a, b). Usually, a CLP1 spiral is of greater dimension than a CLP2 one. All available well-preserved CLP1 spirals, at least eight in number, are well approximated by a single LP with $\kappa = 1.393$ cm approximately. The first LP, together with typical CLP1 spirals best approximated by it, is shown in Figures 3 and 4.

    Spirals depicted on the wall paintings belonging to class LP2 are best approximated by at least two LPs with $\kappa = 0.6263$ cm and $\kappa = 0.2256$ cm. The excellent way the

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Fig. 2. (a) The LP2 class spiral S011. (b) Another LP2 class spiral, the S101.

LP2 prototypes approximate the actual corresponding spirals (the S011 and S101) is demonstrated in Figures 5 and 6.

Notice that the length of the depicted CLP1 spirals varies within certain limits. We would like to emphasize that LP1 constitutes a respectable approximation of

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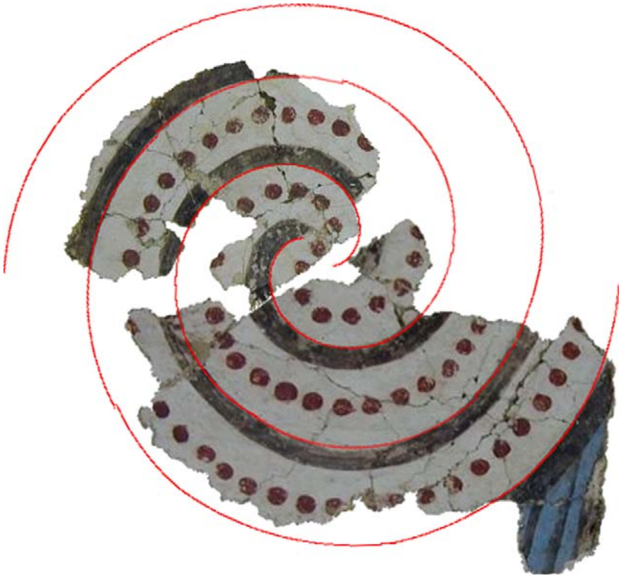

Fig. 3. The excellent way the six stencils (computer-generated red lines) approximate the corresponding spiral.

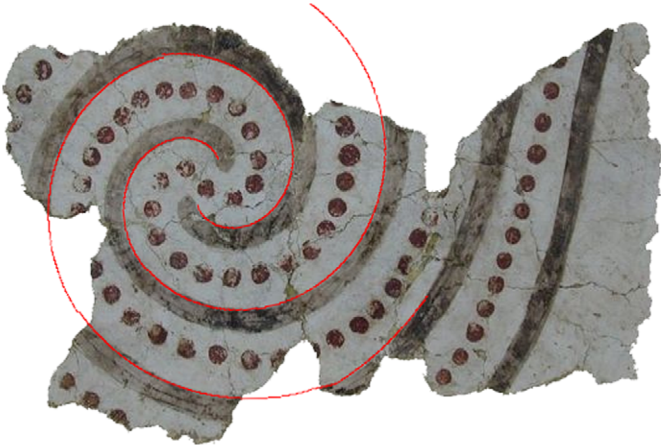

Fig. 4. The excellent way the six stencils (computer-generated red lines) approximate the corresponding spiral.

CLP1 spirals. For example, considering S238 and S022 depicted in Figures 1 and 2, respectively, and using the Euclidian norm, the average distance between the actual and prototype spirals is between 0.14 and 0.20 cm.

LP2 approximates all CLP2 spirals even better. For example, consider the actual spirals S011 and S101, depicted in Figures 5 and 6, respectively, with a radius ranging

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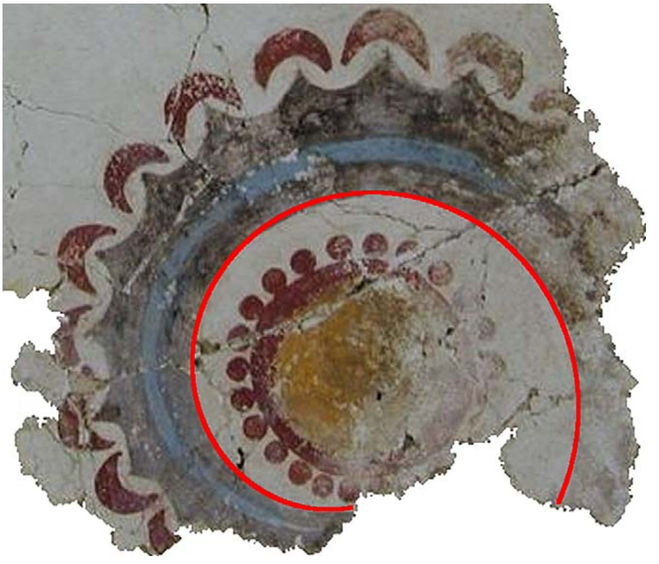

Fig. 5. Demonstration of the excellent way a LP2 spiral with $\kappa = 0.6263$ cm (red) approximates the spiral S011.

Fig. 6. Demonstration of the excellent way the two LP2 spirals with $\kappa = 0.6263$ (red) and $\kappa = 0.2256$cm (green) approximate the pair of spirals S101.

from 4.3 to 8.3 cm approximately, average radius about 6.2 cm, together with the LP2 placed in the best matching position. Then, the average Euclidian distance between this LP2 and these depicted spirals is about 0.08 cm, while the corresponding average percentage error is 1.1% and the maximum one is about 1.6%. Notice that

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the maximum percentage error occurs in the very start of the entire spirals, while the absolute error remains statistically constant throughout the spiral.

(4) The general analysis of the wall paintings, together with the overall archaeological consideration of the Thera civilization of the era, suggests that, most probably, the artist(s) used handicraft stencils or 'French curves' in order to accomplish the wall paintings. Thus, we have examined the possibility that the artist(s) drew the spirals using a number of stencils. Therefore, applying a proper version of the aforementioned original algorithms, we have reached the conclusion that all spiral parts belonging to CLP1 can be exceptionally well approximated by dividing the LP 1 into six parts, thus producing six ideal stencils. In fact, an arbitrary CLP1 spiral part can be approximated by a proper assemblage of these six ideal stencils with an average absolute error of about 0.05 cm, an average relative error of about 0.5%, and maximum percentage error of about 0.8%. These errors are exceptionally low, in practice. In other words, the six LP1 stencils approximate all spiral parts of CLP1 so well that the discrepancy between the model and the actually drawn spirals can be attributed to the digitized image inaccuracies (see Figures 3 and 4).

    LP2 can be divided in two stencils, offering an analogous approximation of CLP2 spirals. However, since the approximation achieved by a single prototype is very good and since the CLP2 dimensions are smaller, one cannot exclude the possibility that CLP2 spirals were drawn by means of a single stencil (see Figures 5 and 6).

What the aforementioned analysis has shown is that the depicted spiral parts match particularly well at least three linear spiral prototypes and exceptionally well if LP1 is divided into a number of stencils. It is clear, however, that, unless a major archaeological find occurs, one cannot decide with certainty about the exact method the artist(s) used to construct the prototypes. Strictly speaking, one cannot exclude that the artist used a kind of elastic rope or thread and unwrapped it around a certain object, so that a figure matching these linear spiral prototypes arose accidentally. However, the fact that at least three different prototypes exist essentially reduces the possibility that the Archimedes spiral models occurred by accident. On the other hand, accepting the hypothesis that these LPs have been formed by means of a geometrical method amounts to recognizing an advanced sense and application of geometry for in this Late Bronze Age civilization.

In any case, we have decided to momentarily accept the aforementioned hypothesis to find out where it can lead. Indeed, the next step, after adopting the hypothesis that the artist generated the linear spiral prototypes using a geometrical method, is to spot at least one such, acceptable for the era, method. Such a method is shown in Figure 7 and can be described as follows:

(1) One draws a large number of homocentric circles Γ_n of common center O, where two consecutive circles have a fixed radius difference, say d.

(2) One divides the 2π entire angle into N equal angles and draws the corresponding semilines ε_n starting at the same center O, where, clearly, ε_{n-1} and ε_n form an angle

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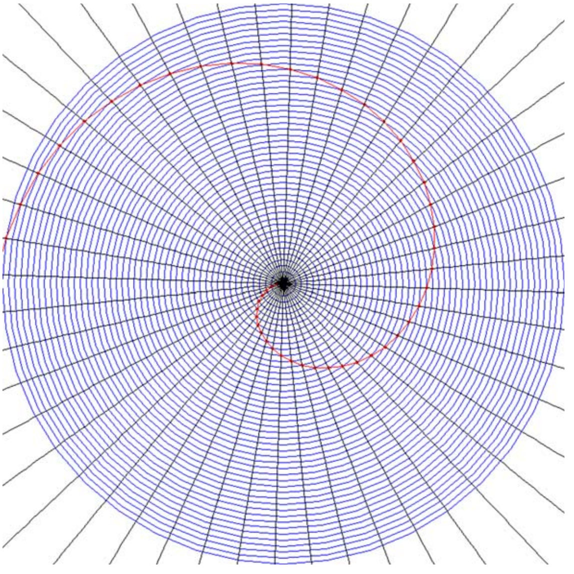

Fig. 7. A geometric method for generating points lying on a linear spiral.

$2\pi/N$. The larger the N, and the smaller the distance d, the better the linear spiral approximation.

(3) The sequence of points of intersection of ε_n and Γ_n lie on an Archimedes spiral.

It is clear that, no matter how simple looking nowadays, this method incorporates a considerable amount of novelty not only for 1650 BC, but even for the Classical Years.

3.2 Knowledge of Drawing Isogonal Lines of Canonical 48- and 32-gon

If the artist indeed used the geometrical method described in the previous section for drawing the linear spiral prototypes, then one may be tempted to investigate the possibility that he/she used a subset of the method for drawing other patterns, as well. Thus, we have considered the red-spot decoration appearing in the class of LP1 spirals (see Figures 1a, b), and we have investigated if there is a sequence of isogonal, concurrent semilines ε_n, upon which the red-spot centers lie. In order to do so, we have developed an original algorithm that uses as parameters the position of the common semilines' center and the angle ϕ between any two successive ε_{n-1} and ε_n. The algorithm converged successfully for all CLP1 spirals, for $\phi = 2\pi/48$. In other words, for each CLP1 spiral, there is a sequence of isogonal semilines corresponding to radii of a canonical 48-gon, upon which all centers, when properly grouped, lie (see Figure 8). In other words, the results of the mathematical analysis fully support the following hypothesis: the artist(s) had a stencil of equiangular lines, with epicenter angle $2\pi/48$, which he/she was properly placing on the

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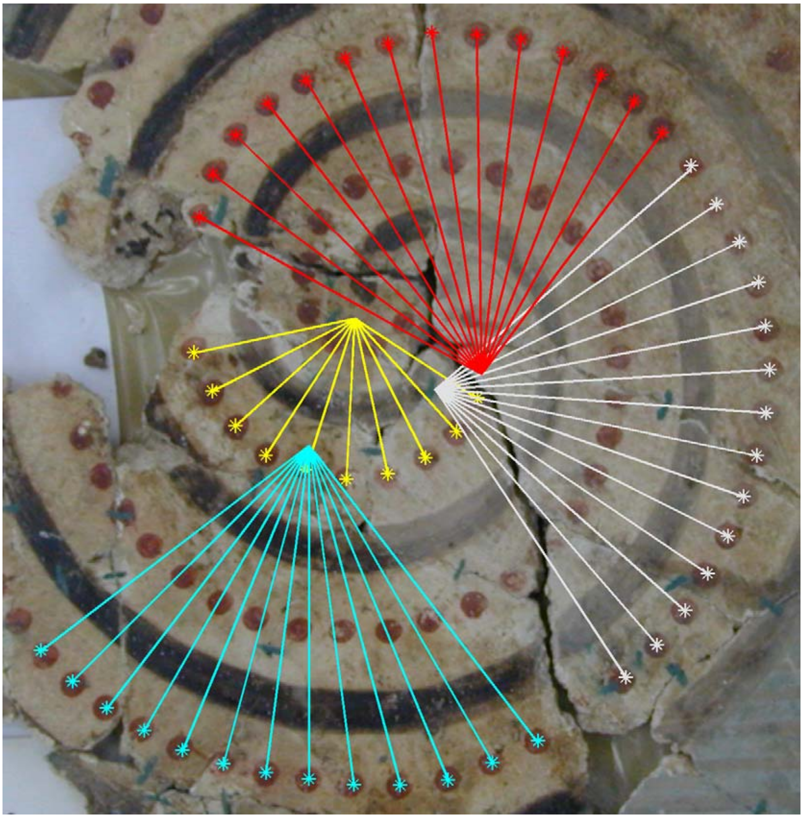

Fig. 8. The centers of the red dots, when properly grouped, lie on the radii of a canonical 48-gon, with exceptionally low angular error. In the first 10-dot group, a radius is skipped each time, most probably for aesthetic reasons.

wall, in order to generate contiguous sets of red dots. In fact, the average discrepancy of each red-spot center from the corresponding semiline ε_n, is less than 0.5 degrees, while the maximum error is less than 0.8 degrees. This discrepancy is exceptionally small, within the range of error with which the red-spot centers are determined. We stress that this best ε_n sequence is unique and that the related common center O is always close to the drawn spiral center and frequently coincides with it (see Figure 8). This fact, clearly, is fully compatible with the hypothesis that the artist drew the Prototype Linear spiral 1 with the aforementioned geometrical method. Moreover, we would like to point out that the distance of two consecutive red-spot centers varies greatly and, thus, one cannot claim that the artist(s) tried to draw red-spots with equidistant centers, and in this way a sequence of red-spot centers that lie on isogonal lines accidentally arose.

Similarly, after adopting the hypothesis that the artist drew the LP2 spirals by means of the same geometrical method, we have once more investigated if a part of the related

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decoration may be a byproduct of this method. Hence, we have investigated if the black spikes pointing at the internal of the tyrian purple crescent-like shapes lie on a sequence of isogonal concurrent semilines ε_n. To test this, we have once more applied the aforementioned original algorithm and thus we have demonstrated that for all CLP2 spirals there is such a sequence ε_n with angle $\phi = 2\pi/32$ between any two successive semilines, and we have demonstrated that practically all these black spikes lie on radii of a regular 32-gon, with center O very close or even at the LP2 center. The O position is again fully compatible with the assumption that the artist drew LP2 via the specific geometrical method. The angular distance of the black spikes from the corresponding ε_n angle is once more quite low: the average distance is 0.45–0.7 degrees, while the maximum is 1.5–2.2 degrees (see Figures 9 and 10).

Clearly, a method for constructing epicenter angles of a canonical 48-gon is to start by drawing an equilateral triangle or a canonical hexagon and continually dichotomizing their epicenter angle, thus generating angles of the canonical 12-, 24-, and 48-gon. Similarly,

Fig. 9. The black spikes lie on isogonal lines corresponding to radii of a canonical 32-gon.

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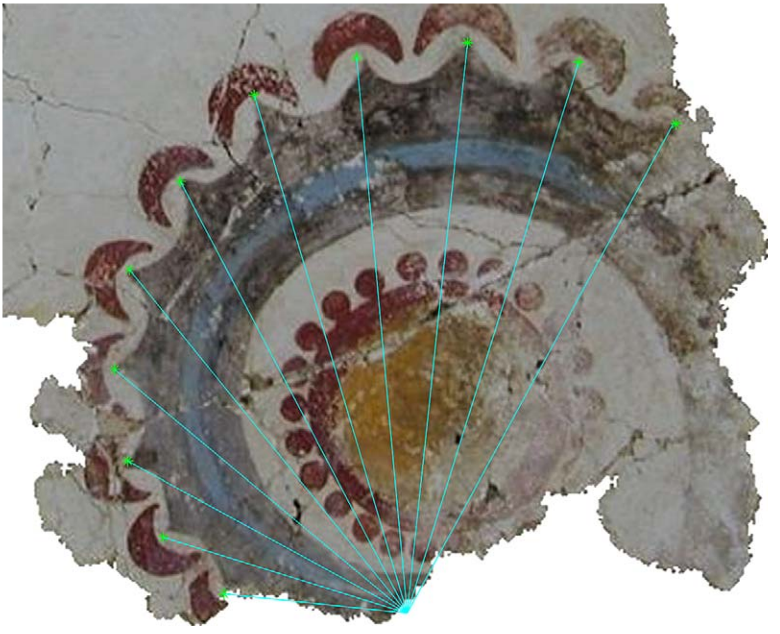

Fig. 10. The black spikes lie on isogonal lines corresponding to radii of a canonical 32-gon.

staring from the diameter of a circle and applying repeated dichotomizations, one may obtain a square, subsequently a canonical 8-, 16-, and 32-gon.

3.3 Knowledge of Drawing Ellipses

A very important, well-designed wall painting excavated in fragments in Akrotiri, at the moment unpublished and not completely restored, depicts myrtles or olive trees. Thorough examination of the well-preserved leaves depicted leads to the conclusion that their borders are particularly smooth, well defined, with essentially steadier line than other wall painting figures. This observation indicates possible use of a tool for drawing the leaves' contour. Since the previous analysis shows that the artist had an arsenal of geometrical methods to choose from, we have checked for a probable geometric shape to which the border leaves correspond. After extensive analysis and related search, we have reached the conclusion that the most probable candidate stencil for the leaves' drawing seemed to be an elliptical one. To test this, we have once more developed a method and corresponding algorithms that find the parameters of the ellipse that best fit each side of all well-preserved available border leaves. In other words, since the polar equations of the ellipse are

$$\begin{aligned} x &= x_0 + \alpha\cos(\theta - \theta_0) \\ y &= y_0 + \beta\sin(\theta - \theta_0) \end{aligned} \tag{2}$$

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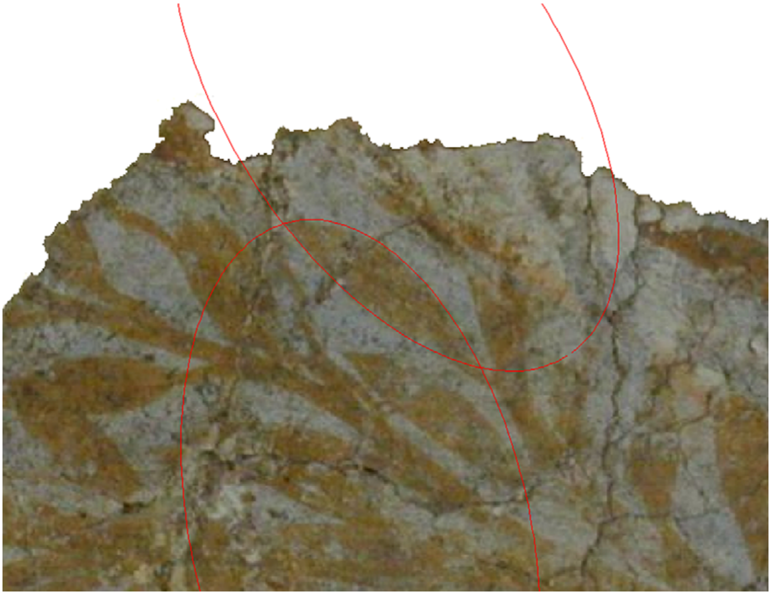

Fig. 11. The way an elliptical stencil could have been used for drawing both sides of a leaf.

we first spot $x_0, y_0, \theta_0, \alpha,$ and β so that the corresponding ellipse best matches a chosen side of the leaf in hand.

After spotting these best parameter sets for all border sides of well-preserved leaves, we have deduced that all these may have been very well generated by two elliptical stencils (see Figure 11). To be specific, there are two prototype ellipses, the first with $\alpha = 4.6$ cm, $\beta = 2.8$ cm and the second with $\alpha = 5.5$ cm, $\beta = 2.97$ cm, which, if properly placed on the wall painting, can approximate all sides (not the stalk) of the 40 available border leaves. The average Euclidian distance of each leaf-border side from the corresponding prototype ellipse part lies in the interval [0.03, 0.06] cm. This is an exceptionally good approximation, which, given the relatively large number of tested borders, as well as the variation in the prototype ellipse placement, strongly supports the hypothesis that the artist indeed used a number of stencils for drawing the leaves' borders. Application of proper statistical criteria indicates that the probability an arbitrary leaf-border side corresponds to one of the aforementioned ellipse stencils is very close to 1.

3.4 Using Hyperbolas, Ellipses, and Linear Spirals for Drawing Complicated Themes

Following the analysis introduced in the previous sections, which concerns the so far unpublished wall paintings of the third floor of Xeste 3, the idea emerged among the authors that the border lines of other wall paintings, like the one depicted in Figure 12, can be

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Fig. 12. A female figure extracted from the celebrated wall painting 'Gathering of Crocus', initially decorating the internal wall of the second floor of Xeste 3.

piecewise approximated by geometric prototype shapes. After developing suitable original algorithms for testing and achieving this approximation, we have reached the conclusion that there are seven geometric archetypes, parts of which can optimally approximate most of the wall paintings' borders initially decorating the internal walls of Xeste 3. These seven geometrical prototypes are one linear spiral, two ellipses, and four hyperbolas, which are shown in Figures 13a–c.

In particular, the geometrical prototypes that were found are the following:

(1) A linear spiral with primary parameter $k = 0.1693$ cm (always depicted in red).
(2) A hyperbola with one of the following four sets of primary parameters: $\left(a_1^{\mathrm{H}}, b_1^{\mathrm{H}}\right) = (14.24\ \mathrm{cm}, 20.12\ \mathrm{cm})$ (magenta), $\left(a_2^{\mathrm{H}}, b_2^{\mathrm{H}}\right) = (7.86\ \mathrm{cm}, 17.63\ \mathrm{cm})$ (cyan), $\left(a_3^{\mathrm{H}}, b_3^{\mathrm{H}}\right) = (4.11\ \mathrm{cm}, 6.29\ \mathrm{cm})$ (green), $\left(a_4^{\mathrm{H}}, b_4^{\mathrm{H}}\right) = (2.09\ \mathrm{cm}, 2.52\ \mathrm{cm})$ (blue).

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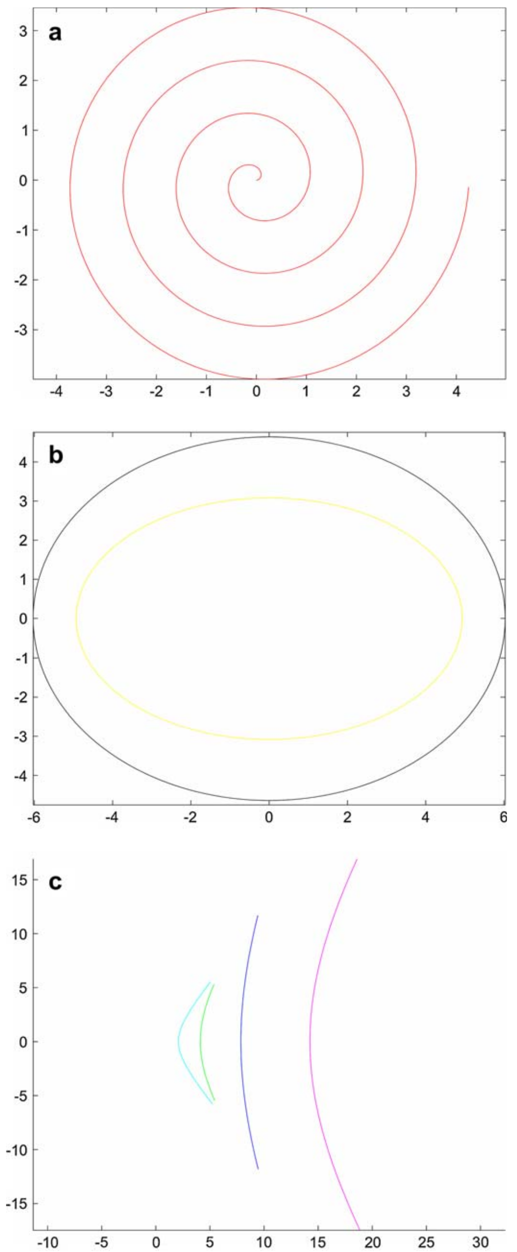

Fig. 13. (a) The linear (Archimedes') spiral archetype used for drawing parts of the wall paintings like the ones depicted in Figures 14 and 15. (b) The two elliptical stencils used for drawing parts of the wall paintings like the ones depicted in Figures 14 and 15. (c) The four hyperbolic stencils used for drawing parts of the wall paintings like the ones depicted in Figures 14 and 15.

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(3) An ellipse with one of the following two sets of primary parameters $\left(a_1^{\mathrm{E}}, b_1^{\mathrm{E}}\right) = (4.64\ \mathrm{cm}, 6.03\ \mathrm{cm})$(yellow), $\left(a_2^{\mathrm{E}}, b_2^{\mathrm{E}}\right) = (3.08\ \mathrm{cm}, 4.93\ \mathrm{cm})$ (purple).

We once more emphasize that each part of the border of the selected wall paintings can be best approximated by only one of the aforementioned geometrical prototypes, shown in Figures 13a–c. The way the geometrical stencils piecewise approximate the corresponding border of the considered set of wall paintings is demonstrated in Figure 14. We note that

Fig. 14. The wall painting shown in Figure 12 with the corresponding geometric prototypes superimposed on its border line parts.

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all lines of the depicted figures are approximated by the obtained set of stencils with an exceptionally low average error of $3 \cdot 10^{-4}$ m.

In addition, the fact that the various parts of the borders are approximated by the seven aforementioned geometric curves (stencils) with impressive precision strongly indicates that there was a very neat and precise method of drawing these geometric figures and constructing the corresponding stencils. One cannot exclude that this method of drawing these curves was a geometric one. However, it is practically impossible to draw these particular curves by hand at random.

4. *General Remarks and Conclusions*

In the previous analysis, we have given strong evidence that a person or a group of persons were capable of (1) dividing the entire 2π radians angle into 16, 32, and 48 equal angles, (2) constructing linear (Archimedes') spirals, and (3) drawing ellipses and hyperbolas, all with remarkable precision. In connection to these findings, we would like to make the following remarks:

One can make several hypotheses about the methods and tools the artist(s) used to draw the corresponding figures. However, unless a major archaeological find takes place, it is extremely difficult to decide with certainty the exact form of these methods. Still, the remarkable accuracy with which each prototype geometrical configuration (the linear spiral, the sequence of isogonal semilines, or the ellipse and hyperbolas) approximates the corresponding drawing strongly supports the conjecture that the artist(s) used geometrical methods to construct the archetypes. Therefore, we will risk making a number of hypotheses: the fact that the artist(s) knew how to construct such a variety of isogonal straight lines, like the ones depicted in Figures 8–10, implies the idea that he/she had a method of dichotomizing an angle. If he/she just knew how to divide the 2π radians angle into 4, 8, or even 16 equal angles, then one may assume that he/she constructed them by successive anadiploses (folding) of a circle. We once more stress that Thales seems to be the first who stated that the circle is bisected by its diameter, 1150 years later. However, the existence of angles corresponding to the epicenter angle of a canonical 32-gon and, in particular, of a canonical 48-gon in the wall paintings indicates that he probably used a more complicated geometrical method.

Concerning spirals, the fact that there are at least two classes, approximated by three linear spiral prototypes with essentially different sets of parameters, clearly strengthens the hypothesis that the spirals were drawn not accidentally but via a concrete method. One can perhaps figure out a number of tools for drawing a linear spiral, including one using a kind of guided unwrapping. On the other hand, the accuracy with which the drawn spirals match the prototypes supports the hypothesis that the prototypes were constructed by means of a strict geometrical method. In other words, if the stated hypothesis is correct, the artist(s) applied a successive dichotomization of epicenter angles in order to generate

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a dense sequence of points lying on a linear spiral and then joined them properly to form the corresponding continuous curve.

Moreover, the performed mathematical and computational processing by the authors suggests that the artist probably divided the spiral prototypes into a number of stencils and then he/she used them to reproduce the spirals on the frescos.

In a similar manner, the fact that more than one elliptical and hyperbolic geometrical prototypes were spotted in the drawings of myrtle or olive leaves and the lady's wall painting (see Figures 11 and 14) strongly suggests that the artist had a concrete method of constructing these prototypes as well. We would like to point out that it is practically impossible to generate four different hyperbolas with such impressive accuracy, at random and/or by freehand.

Moreover, we must emphasize the remarkable fact that the artist(s) used a small number of geometric tools to draw an impressive variety of figures, such as female and male bodies and faces, animals (birds, fish), clothes and their designs, jewelry, plants, household objects, mythical figures. For example, a single hyperbolic stencil was used to draw borderlines of more than one part of the human body; thus, for example, the same hyperbolic stencil was used to draw a hunch, both outlines of the forearm, a thigh, an abdominal region, etc. A small, for space economy reasons, assortment of such drawings, together with their corresponding stencils, is shown in Figure 15. The aforementioned drawing methodology indicates an impressive sense of geometry, symmetry, analogy/proportionality, and fascinating imagination. To the best of our knowledge, the use of geometric stencils for achieving high-quality figure representation seems to be unique in the History of Classical Arts and Sciences.

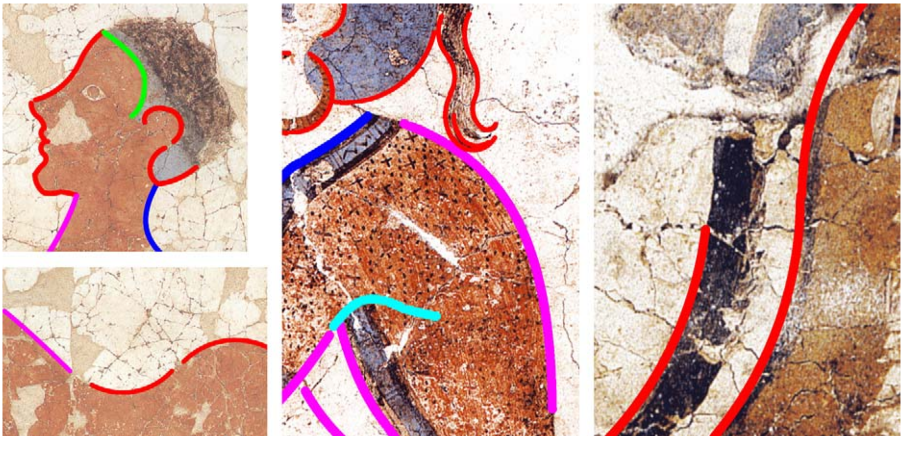

Fig. 15. Parts of other wall paintings with the corresponding geometric prototypes superimposed on them.

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In any case, it seems that there was an impressive accumulation of geometrical knowledge in a group of persons in the prehistoric civilization of Akrotiri, Thera, around 1650 BC. In congruity with Hoyrup (1996), we feel that one must not see this knowledge from the mathematical point of view only; on the contrary, one must consider it integrated into the general historical and cultural framework of the specific civilization. For example, from the archaeological point of view, it is probable that the wall paintings with geometrical representations were associated with a kind of mysticism pertaining to this prehistoric society.

For the generation and accumulation of the geometrical sense and skill expressed in the Thera wall paintings, it seems that a considerable amount of original ideas was necessary, not directly associated with measurements and fractions. Although one cannot be certain of the birthplace of these original ideas, the fact that no other drawings or representations with such impressively accurate approximation of the corresponding geometrical configurations have been found elsewhere so far strongly supports the hypothesis that these novel ideas were generated in an Aegean (Akrotiri / Thera? Crete?) civilization. However, even from the archaeological point of view, it seems utopia to believe that there is no continuous flow of information between civilizations.

On the other hand, one must not underestimate the role of original ideas generated by a person or a group of persons. The original ideas, that often make use of existing knowledge, lead to an expansion of the related gnosiological field. If the degree of originality-novelty is high, then the resulting gnosiological domain may be substantially different from the previous one, even if the former has made use of the latter.

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