Elastic-plastic response spectra for exponential blast loading

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Abstract
The design of structures subjected to loads due to explosions is often treated by means of elastic-plastic response spectra. Such spectra that are currently available in the literature were computed on the basis of triangular shape of blast pressure with respect to time, and by neglecting the unloading stages of the structural response. In the present paper, response spectra based on a more accurate exponential distribution of blast pressure, and accounting for all stages and cycles of response, are proposed. To that effect, analytical expressions of the solutions of the pertinent equations of motion have been obtained via symbolic manipulation software, and have been used to carry out an extensive parametric study. A comparison of the spectra obtained by the proposed approach to the existing ones, reveals that the commonly used assumption of triangular blast load evolution with time can sometimes be slightly unconservative, particularly for flexible structural systems, but can also be significantly overconservative for stiffer structures.

Keywords
blast loading, elastic-plastic response spectra, exponential load distribution, structural design, dynamic analysis

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1. Introduction

Structures may experience blast loads due to military actions, accidental explosions or terrorist activities. Such loads may cause severe damage or collapse due to their high intensity, dynamic nature, and usually different direction compared to common design loads. Collapse of one structural member in the vicinity of the source of explosion, may then create critical stress redistributions and lead to collapse of other members and, eventually, of the whole structure. A recent example of such a failure was the well-known collapse of the Alfred P. Murrah Federal Building in Oklahoma City, Oklahoma following a terrorist attack (Prendergast [1], Sozen et al. [2]).

For some structures blast resistant design may be required, if their use is such that there is a high risk for such a loading incident to be encountered. Examples include government buildings, constructed facilities in petrochemical plants or bunkers in military installations. For such structures it is desirable to establish design procedures and construction techniques necessary to achieve the required strength to resist the applied blast loads (Rittenhouse [3], Ettourney et al. [4]).

This problem can be tackled in several different ways. The approach that more accurately describes the dynamic response of structures to explosive loads is via numerical analysis, usually by means of the finite element method. Such analyses can capture the geometry of the structure, the spatial and temporal distribution of the applied blast pressure, as well as the effects of material and geometric nonlinearity, in a satisfactory manner, and have been performed by several investigators.

An example of this type is the work of Louca et al. [5] who presented results of the response of a typical blast wall and a tee-stiffened panel subjected to hydrocarbon explosions with geometries typical of those used in offshore structures. They performed nonlinear finite element analyses, accounting for the effects of plasticity, strain-rate and buckling. They also compared their results with experimental data as well as approximate solutions obtained with a single degree-of-freedom model. Other investigators have performed similar analyses to point out the importance of detailing, for example Otani and Krauthammer [6], who performed nonlinear dynamic finite element analysis of typical thre-dimensional slab-to-slab-to slab joints in order to investigate the effect of reinforcement details.
However, such analyses require highly specialized software, elaborate calibration of the models with experimental (Krauthammer et al. [7]) or other established results, and extensive computational effort. Therefore, this approach is used, for the time being, primarily for research purposes. Practice oriented design, assessment and protection methods do take advantage of advanced numerical analysis, but focus also on engineering detailing, connections and conceptual design (ASCE [8], Weidlinger Associates web page [9]).

An alternative design approach, which involves several approximations, but is rather easily applicable in routine design, is recommended by the U.S. Department of the Army TM5-1300, [10], and has been adopted by other researchers, for example Mays and Smith [11]. It is based upon substituting the structural element by a stiffness equivalent, single degree-of-freedom structural system, and using elastic-plastic response spectra to predict the maximum response of this system.

The response spectra accompanying the above methodology in the literature have been computed via numerical integration of the equations of motion, assuming triangular loading evolution with time, and applying the average acceleration method. Furthermore, these spectra do not take into account the fact that the response consists of several stages and cycles, and that the equations of motion change in each stage.

These simplifying assumptions are acceptable and provide satisfactory results in most cases, considering also all other uncertainties that are inherent in this design approach, such as material modeling, substitution of the real structure by a single degree-of-freedom (SDOF) model, and explosion characteristics. Nevertheless, the fact remains that the actual time variation of blast pressure can be approximated much better by an exponential function, and that in some cases the maximum response may be encountered not in the first stage of response but in a subsequent one.

The objective of this paper is to assess the importance of these assumptions for practical design, and to provide a more accurate design tool, without sacrificing ease of use for the practicing engineer. This is achieved by deriving the equations of motion based on a more accurate exponential distribution of blast pressure, and accounting for all stages and cycles of response, obtaining analytical expressions of the solutions via symbolic manipulation software, and using these solutions to carry out an extensive parametric study, and draw a new set of response spectra. A comparison of these spectra to the existing ones reveals that
the commonly used assumption of triangular blast load evolution with time can sometimes be slightly unconservative, particularly for flexible structural systems, but can also be significantly overconservative for stiffer structures.

2. **Idealization of blast loading**

Chemical investigation and experimental data have shown that the evolution of blast load pressure $P$ with time $t$ can be simulated rather accurately by an exponential distribution, which has a start peak pressure $P_s$, as shown by the continuous line in figure 1 and described by the following equation (Baker et al [12], Bangash [13]):

$$P(t) = P_s \left(1 - \frac{t}{t_d}\right) e^{-bt}$$

In equation (1), $t_d$ is the time of reversal of direction of pressure, and $b$ is a shape parameter depending on the dimensionless scaled distance $Z$, usually taken equal to 1. The scaled distance $Z$ is given by

$$Z = \frac{R}{3W}$$

where $R$ is the distance from the center of a spherical charge in meters and $W$ is the charge mass expressed in kilograms of equivalent TNT. Several blast wave parameters, such as peak static over-pressure, time duration of positive phase and the impulse of the wave are plotted with respect to $Z$ in a number of references.

The usual simple way to simulate a blast load for structural analysis and design purposes, is with a triangular distribution, which has a start peak pressure $P_s$ and decreases linearly with time within a time period $t_d$, as illustrated by the dotted line in figure 1, and described by the equation

$$P(t) = \begin{cases} 
P_s \left(1 - \frac{t}{t_d}\right), & t < t_d \\
0, & t > t_d
\end{cases}$$

(3)
3. **Static and dynamic elastic-plastic behavior for SDOF systems**

An elastic-perfectly plastic SDOF system without damping has the static resistance function $R$, shown in figure 2, where resistance is plotted in the vertical axis with respect to the degree-of-freedom $x$, plotted in the horizontal axis. The response of the system is divided in four stages, as described for example by Bangash [13] and Chopra [14]:

1. Response up to the elastic limit $x_{el}$, corresponding to a maximum resistance $R_m$, characterized by the elastic stiffness $k$ of the system, defined as the ratio of $R_m$ to $x_{el}$.

2. Plastic response with constant resistance $R_m$ from the elastic limit $x_{el}$ up to a maximum displacement $x_m$.

3. Unloading, where the response starts to decrease with the same absolute stiffness as in stage (1) up to a maximum (in absolute terms) negative resistance $-R_m$.

4. Plastic response during unloading, corresponding to a resistance $-R_m$.

The analytical expressions describing the response in each stage are:

$$R = \begin{cases} 
  kx, & 0 < x < x_{el} \\
  R_m, & x_{el} < x < x_m \\
  R_m - k(x_{el} - x), & x_m - 2x_{el} < x < x_m \\
  -R_m, & x < x_m - 2x_{el}
\end{cases} \quad (stage \ 1)$$

The response of such a system without damping subjected to dynamic excitation $P(t)$ is described by the following equations of motion:

**Stage (1)**

In this stage the equation describing the response of the system is:
\[ m\ddot{x}(t) + kx(t) = P(t) \]  \hspace{1cm} (5)

The solution \( x(t) \) of equation (5) depends on the load function \( P(t) \). The initial conditions are usually zero. The typical profile of the displacement in the first stage can be seen in figures 3a and 5a, corresponding to the cases when the system reaches plasticity (stage 2) before or after elastic rebounding, respectively. If \( t_{el} \) is the time at the end of the elastic region, where the response is \( x_{el} \), then we have the following displacement and velocity at the end of stage (1):

\[
\begin{align*}
    x(t_{el}) &= x_{el} \\
    v_{el} &= \dot{x}(t_{el})
\end{align*}
\]  \hspace{1cm} (6)

The time \( t_{el} \) is obtained by solving the first of equations (6), as we know the displacement \( x_{el} \) from the static resistance function of the SDOF system, given by figure 2 and equation (4). Then, \( v_{el} \) is calculated from the second of equations (6). The displacement \( x_{el} \) and the velocity \( v_{el} \) at the end of this stage are the initial conditions for the second stage.

**Stage (2)**

In this stage the system has reached plasticity, and the equation that describes its response is:

\[ m\ddot{x}(t) \pm R_m = P(t) \]  \hspace{1cm} (7)

The sign of \( R_m \) in equation (7) depends on whether the system comes to plasticity (stage 2) before elastic rebounding (figure 3), in which case (-) is used, or after elastic rebounding (figure 5), when (+) is used. The solution \( x(t) \) of equation (7) depends again on the load function \( P(t) \) and the initial conditions \( x_{el} \) and \( v_{el} \). The displacement in the second stage can be seen in figures 3b and 5b, respectively, between the time \( t_{el} \), corresponding to the first occurrence of plasticity, and the time \( t_m \) of maximum response.

By setting the velocity in the second stage, given by the derivative of the solution of equation (7), equal to zero, the time of maximum response \( t_m \) is obtained.

\[ \dot{x}(t_m) = 0 \]  \hspace{1cm} (8)

Then, the maximum response \( x_m \) can be obtained replacing the time \( t \) with \( t_m \) in the solution \( x(t) \) of equation (7).
\[ x_m = x(t_m) \]  \hspace{1cm} (9)

The new displacement \( x_m \), and the corresponding velocity \( v_m = 0 \) are the initial conditions for the next stage (3).

**Stages (3) and (4)**

A similar procedure is followed to obtain the response in the next stages (3) and (4). Graphically we can see these stages in figures 3c and 3d for the case when the system reaches plasticity (stage 2) before elastic rebounding. The time of transition from stage (3) to stage (4) is denoted by \( t_{reb} \), and the maximum displacement in stage (4) is named \( x'_m \) and the corresponding time \( t'_m \).

When the fourth stage finishes, then the second cycle starts and four new stages are possible to occur. The response of the second cycle can be treated with the same procedure. In figures 3e-3h these four stages of the new cycle can be seen, where the maximum displacement and the corresponding time in the second and fourth stage are named \( x_{m2} \), \( x'_{m2} \) and \( t_{m2} \), \( t'_{m2} \), respectively. In figure 4 the total response of system in the two cycles is shown. The absolute maximum displacement may occur either in the second or in the fourth stage of each cycle, in other words the maximum displacement of the system could be anyone of \( x_m, x'_m, x_{m2} \) and \( x'_{m2} \). The number of cycles depends on the ratio of maximum system resistance to external load \( R_m/P \). As this ratio increases, the cycles decrease until the ratio becomes equal to 2, in which case there is no cycle and the system behaves elastically.

When plasticity occurs after elastic rebounding, the system usually works elastically after the third stage of the first cycle and does not enter the plasticity region again (figure 5c). As can be observed from figure 6, where the total response is presented for that case, the maximum displacement could be either \( x_m \) or \( x'_m \), corresponding to first plasticity and to the elastic region, respectively.

Because of the shape of the loading function \( P(t) \) in the case of explosions, the maximum response of the system is obtained in most cases from the first cycle and only rarely from the second or third cycle, as could be the case if we had a harmonic loading function \( P(t) \). This will be illustrated better in the following sections.
4. **Solutions of equations of motion of SDOF systems under explosive loads**

The solutions of the dynamic equations of motion (5) and (7) of a single-degree-of-freedom system subjected to explosive loads have been obtained analytically, using the symbolic manipulation software Mathematica, Wolfram [15]. Both cases of exponential and triangular evolution of the loading function with time have been considered, as described by equations (1) and (3), respectively. The resulting solutions for the different stages of response are the following:

### 4.1 Triangular loading

#### Stage (1)

\[
\ddot{x}(t) = \frac{2\pi(t_d - t) - 2\pi t_d \cos \frac{2\pi t}{T} + T \sin \frac{2\pi t}{T}}{2\pi t_d}
\]  

where:

\[
\dot{x}(t) = \frac{x(t)}{x_{el}}, \quad a = \frac{R_m}{P_s}
\]

#### Stage (2)

**Case I:** \(t_{el} < t_d\)

a) For \(t_{el} < t < t_d\), the equation of motion is \(m \ddot{x}(t) + R_m = P(t)\), and its solution is given by:

\[
\ddot{x}(t) = \frac{1}{6\pi^2 t_d^2} \left\{ -2\pi \left( 2\pi^2 t^3 + 3\pi t^2 t_d^2 + 6\pi^2 t^2 t_d^2 - 3\pi^2 t_d^2 + 12\pi^2 t_d t_{el} - 6\pi^2 t_{el}^2 - 6\pi^2 t_{el}^2 + 6\pi^2 t_d t_{el} + 4\pi^2 t_{el}^3 \right) + 6\pi T^2 \left( t - t_d + t_{el} \right) \cos \frac{2\pi t_{el}}{T} + 3T \left( T^2 + 4\pi^2 t_d^2 - 4\pi^2 t_{el}^2 \right) \sin \frac{2\pi t_{el}}{T} \right\}
\]

b) For \(t_{el} < t_d < t\), the equation of motion is \(m \ddot{x}(t) + R_m = 0\), and the response is obtained as:
\[ x(t) = \frac{1}{6\alpha T^2 t_d} \left\{ -2\pi \left( 3tT^2 + 6\alpha t^3 t_d + 3T^2 t_d + 6\pi^2 t_d^3 + 12\alpha t^3 t_d + 12\alpha \pi^2 t_d t_t - 12\alpha \pi^2 t_d t_t \right) \cos \frac{2\pi t_d}{T} + \right. \]
\[ \left. 6\pi^2 t_d^3 + 6\pi^2 t_d^3 + 6\pi^2 t_d^3 + 6\pi T^2 \left( t - t_d - t_d \right) \sin \frac{2\pi t_d}{T} \right\} + \]
\[ 3T \left( T^2 + 4\pi^2 t_d^3 - 4\pi^2 t_d t_t \right) \sin \frac{2\pi t_d}{T} \] \hfill (13)

Case II: \( t_d < t_e \)

a) For \( t_d < t < t_e \), the equation of motion is \( m \ddot{x}(t) + kx(t) = 0 \), and the solution is:

\[ x(t) = \frac{2\pi t_d \cos \frac{2\pi t}{T} - T \sin \frac{2\pi t}{T} + \sin \frac{2\pi (t - t_d)}{T}}{2anT_d} \] \hfill (14)

b) For \( t_d < t_e < t \), the equation of motion is \( m \ddot{x}(t) + R_m = 0 \), and its solution is given by:

\[ x(t) = \frac{1}{2anT^2 t_d} \left\{ -4\alpha \pi^2 t_d + 8\alpha \pi^3 t_d t_e - 4\alpha \pi^3 t_d t_e - 2\pi T^2 \left( t - t_d \right) \cos \frac{2\pi (t - t_d)}{T} + \right. \]
\[ \left. 2\pi T^2 \left( t - t_d - t_e \right) \cos \frac{2\pi t_d}{T} + T^3 \sin \frac{2\pi t_d}{T} + T^3 \sin \frac{2\pi t_d}{T} + \right. \]
\[ \left. 4\pi^2 T t_d \sin \frac{2\pi t_d}{T} - 4\pi^2 T t_d \sin \frac{2\pi t_d}{T} \right\} \] \hfill (15)

The analytical solutions for stages (3) and (4) are obtained in a similar manner.

### 4.2 Exponential loading

**Stage (1)**

\[ x(t) = \frac{1}{a(T^2 + 4\pi^2 t_d^2)^2} \left\{ 4e^{\frac{t}{T}} \pi^2 t_d \left( t T^2 + T^2 t_d + 4\pi^2 t_d^3 - 4\pi^2 t_d^3 + \right. \right. \]
\[ \left. \left. e^{\frac{t}{T}} t_d \left( -T^2 + 4\pi^2 t_d^3 \right) \cos \frac{2\pi}{T} - 4e^{\frac{t}{T}} \pi T t_d^3 \sin \frac{2\pi}{T} \right\} \] \hfill (16)
Stage (2)

\[
\ddot{x}(t) = \frac{1}{a(T^3 + 4\pi^2 T d^2)^2} \left\{ 2\pi \left( T^2 + 4\pi^2 T d^2 \right)^2 \left[ \frac{t}{T d} \left( t + T d \right) - a \left( t - t_{el} \right)^2 \right] + \right.
\]

\[
8e^{\frac{-t}{T d}} \pi^2 T \left[ 4\pi^2 T d \left( t^2 + t_{d} t_{el} + t_{el}^2 \right) + T^2 \left( 3t_{d}^2 + 3t_{d} t_{el} + t_{el}^2 \right) - 
\]

\[
t \left( 4\pi^2 T d t_{el} + T^2 \left( 2t_{d} + t_{el} \right) \right) \right] + 
\]

\[
2T d \left[ T \left( T^2 - 4\pi^2 T d \left( -2t + t_{d} + 2t_{el} \right) \right) \cos \frac{2\pi t_{el}}{T} + 
\]

\[
2\pi \left( -tT^2 + 2T^2 T d + 4\pi^2 T d^2 t + T t_{el} - 4\pi^2 T d^2 t_{el} \right) \sin \frac{2\pi t_{el}}{T} \right\}
\]

The analytical solutions for stages (3) and (4) are obtained in a similar manner.

5. **Elastic response of SDOF systems to explosive loads**

As long as the system remains elastic, the main response parameter is the ratio of the characteristic blast loading duration \(t_d\) to the system’s fundamental period \(T\). A typical elastic response is shown in figure 7, for the case of exponential blast loading, if \(T/t_d=0.1\). The influence of \(T/t_d\) is illustrated very well in the 3D plot of figure 8, showing elastic response to exponential blast load, for varying values of the \(T/t_d\) ratio. Figure 9 shows an elastic response spectrum, containing the maximum response for different values of \(T/t_d\).

6. **Elastic-plastic response spectra**

The elastic analysis described briefly in section 5, is not suitable for the design of structures, which are in the vicinity of an explosion source, as it is highly unlikely that such structures will respond elastically. Therefore, elastic-plastic design should be applied. Given the
dynamic nature of the response, but also the uncertainties in the characteristics of the blast loading, a response spectrum analysis seems to be appropriate.

To that effect, the elastic-plastic analysis described in section 3, has been performed for a characteristic blast loading duration of $t_d=0.1\,\text{sec}$, and a wide range of fundamental periods of the SDOF system and range of ratios $R_m/P$. The maximum values of displacement ratios $\mu=x_{\text{max}}/x_{\text{el}}$ ($x_{\text{max}}=\max\{x_m, x_{m2}, x'_m, x'_{m2}\}$) have been plotted as response spectra for both triangular and exponential blast loading, and are shown in figures 10 and 11, respectively. In addition, the time $t_{\text{max}}$ of occurrence of maximum displacement has been plotted for both loading assumptions, and is illustrated in figures 12 and 13.

The objectives of this analysis were the following:

i. To compare the response spectra for triangular loading to those found in the literature (U.S. Department of the Army Technical Manual), and thus verify the proposed approach, and assess the influence of neglecting the possible occurrence of maximum response in subsequent responses stages beyond the first.

ii. To compare the response spectra for triangular loading to those for exponential one, in order to assess the influence of the assumption of triangular distribution on practical design problems.

iii. To draw qualitative conclusions regarding the response of structures subjected to explosive loadings.

iv. To provide designers of structures subjected to explosions with a more accurate design tool than the one currently in use.

First it should be noted that, in spite of the nonlinearity of the system, it is possible to use nondimensional parameters in order to describe the response. It is the ratios $R_m/P$ and $t_d/T$ that determine the response, and not the actual values of these four parameters.

In order to meet the first two objectives, characteristic values of maximum response are listed in table 1 for different combinations of ratios $R_m/P$ and $t_d/T$. The first observation is that there are some small differences between our results for triangular loading and those given in the U.S. Department of the Army Technical Manual TM5-1300 [10]. This is attributed to the fact that the proposed analysis considers analytical expressions for several cycles of response and
four stages in each cycle, while the analysis in TM5-1300 used the average acceleration method and did not take into account the fact that the equations of motion change in each stage.

The smaller the ratio $R_m/P$, the larger difference is observed in $\mu=x_{\text{max}}/x_{\text{el}}$. The explanation is that for low $R_m/P$ ratios, ranging between 0.3 and 0.8, the maximum response was obtained from the fourth stage, something that was ignored in TM5-1300. Furthermore, for very low $R_m/P$ ratios, ranging between 0.1 and 0.3, the maximum response occurred in the second cycle, which was also not taken into account in the analysis of TM5-1300. In cases of $R_m/P$ ratios higher than 0.8, the maximum response usually was encountered in the first or second stage of the first cycle.

Comparing also the results for exponential loading with those for triangular one, it is noted that differences do occur, which are significant in some cases. When the ratio $\mu=x_{\text{max}}/x_{\text{el}}$ is large, ranging between 1 and 2, the response for exponential loading is less compared to the one for triangular loading. This can be explained qualitatively considering the shape of loading. The exponential loading decreases faster than the triangular one, and this has more influence in elastic-plastic situations than in purely elastic ones.

Some exceptions to this were encountered in case of low $R_m/P$ ratios, for example for $R_m/P=0.1$ and $t_d/T=0.4$, where the exponential loading gives higher response than the triangular one. This happens in cases when the maximum response is obtained at times larger than $t_d$, and the exponential loading has changed sign and may be in phase with the system motion, while the triangular loading at this time is zero.

Figure 14 shows a case for which significant differences between response to triangular and exponential loading were observed, particularly for rather large values of $t_d/T$. Considering that the duration $t_d$ usually varies within a small range of values, large values of $t_d/T$ correspond to small values of the fundamental period $T$, therefore, to stiff systems.

Summarizing the results presented in table 1 and figure 14, it can be said, that the commonly used assumption of triangular blast load evolution with time can sometimes be slightly unconservative, particularly for flexible structural systems, but can also be significantly overconservative for stiffer structures.
A common observation for both exponential and triangular loading is than when the ratio $R_m/P$ is greater than 1.5, the system behaves elastically and no significant difference in the response is encountered.

A comment found in the literature for example by Chopra [14] was verified by the present analysis, namely that in elastic response spectrum calculations, if the time duration $t_d/T$ is longer than 0.5, then the maximum deformation occurs during the pulse and the shape of loading is of great significance. If on the other hand, the time duration $t_d/T$ is less than 0.5, then the maximum deformation occurs during the free vibration and is mainly controlled by the time integral of the pulse, independently of the shape of loading.

7. **Design implications**

Following the design procedure suggested by Mays and Smith [11] for a structural member in flexure, the following steps must be followed:

1. Carry out a preliminary design assuming an equivalent static ultimate resistance and appropriate deformation limits for the desired protection category.

2. Calculate the natural period of the element.

3. Refer to the appropriate SDOF response spectrum for an elastic-plastic system to obtain the maximum response and compare this response to the desired deformation limits.

It is apparent that the third step is directly dependent upon the response spectrum to be used, therefore, the proposed design approach will be erroneous to the extent that the spectrum is inaccurate. Hence, in cases of significant discrepancies between existing and proposed response spectra, the proposed approach will have important design implications.

8. **Summary and conclusions**

The response of structures subjected to loads due to explosions has been investigated. The preliminary and potentially also the final design of such structures, as well as the assessment
of the bearing capacity of existing structures, is often treated by means of elastic-plastic response spectra. Such spectra that are currently available in the literature are based on triangular time evolution of the blast pressure, and neglect the possibility that the maximum response may be encountered during the unloading stages. In the present paper, response spectra based on a more accurate exponential distribution of blast pressure, and accounting for all stages and cycles of response, have been computed. This has been achieved by deriving the pertinent equations of motion, and obtaining analytical expressions of their solutions via symbolic manipulation software. A comparison of the spectra obtained by the proposed approach to the ones for triangular pressure function, led to the conclusion that design based on the commonly used assumption of triangular blast load evolution can sometimes be slightly unconservative, particularly for flexible structural systems, but can also be significantly overconservative for stiffer structures.

References


