Critical Point Detection Using the Length Ratio (LR) for Line Generalization

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Abstract

This article presents a review of existing methods for the detection of critical points cited in cartographic and computer science literature. Moreover, a theoretical assessment of algorithm validity with regard to cartographic representation demands is carried out. A method for the detection of critical points using the length ratio (LR) index is introduced, based on geometric principles. Four lines (three geomorphologic and one arbitrary) selected from relevant studies are used to check the method. Finally, the LR index is used to compare the results of two line simplification algorithms (*pointremove* and *bendsimplify*) applied on five successive line simplification tasks on the coastline of a small island.

Keywords: critical/dominant points, line generalization, line simplification algorithms

Résumé

Dans cet article, on passe en revue les diverses méthodes de détection des points critiques souvent citées en cartographie et en informatique. De plus, on effectue une évaluation théorique de la validité des algorithmes pour ce qui est des exigences en matière de représentation cartographique. Une méthode de détection des points critiques à l'aide d'un indice du rapport des longueurs est présentée selon les principes géométriques. Quatre lignes (trois géomorphologiques et une arbitraire) provenant d'études pertinentes permettent de vérifier cette méthode. Enfin, l'indice du rapport des longueurs est employé pour comparer les résultats de deux algorithmes de simplification de lignes (méthodes de simplification des formes *bendsimplify* et de réduction du nombre de points *pointremove*) utilisés lors de cinq tâches successives de simplification des lignes correspondant à la côte d'une petite île.

Mots clés: points critiques/dominants, généralisation cartographique, algorithmes de simplification de lignes

Introduction

In a study to clarify several concepts of visual perception in the context of information theory, Fred Attneave (1954) mentions that during the observation of an object, the human brain receives a great number of stimuli, which transfer a similarly large amount of information that, in general, cannot be stored and utilized. But humans require only a small part of this information to recognize objects as specific individual entities. The human brain acts in a number of ways (conscious or not) that remove redundant stimuli, after which the entrant information is described and encoded up to the available storage capacity. Among others, Attneave assumes that critical information is concentrated along contours and especially at those points where the slope changes most rapidly. Attneave points out that "common objects may be represented with great economy, and fairly striking fidelity, by copying the points at which their contours change direction maximally, and then connecting these points appropriately with a straight-edge" (1954, 185). He concludes that the locations that configure the shape of a graphic object can be classified quantitatively according to the information they transmit to the observer. Locations that convey a large amount of information characteristically are capable of describing the shape of an object. This principle, articulated by Attneave in 1954, became a research topic in cartography as well as in the domain of computer science. In the cartographic literature, characteristic points are usually termed "critical points"; the term "dominant points" is used in computer science literature. In his definition of critical points, Herbert Freeman (1978) includes those points of a digital line that are (1) curvature maxima and minima, (2) open line end points, (3) points of intersection, (4) points of inflection, (5) points of tangency, and (6) discontinuities in curvature. In addition, critical points found as maxima, minima and zeroes of curvature are invariant under rotations, translations, and uniform scaling (Hoffman and Richards 1982).

In cartography, the concept of critical points refers to generalization and especially to line simplification. Jill Marino (1979) underlines the existence and impact of critical points in an interesting empirical study. In her research, six lines representing natural phenomena with various morphological characters were presented to a group of cartographers and non-cartographers. Study participants were asked to select a set of points that they considered to be necessary and sufficient to retain the character of the line. The statistical analysis of the results led to three basic conclusions: (1) there is a close agreement among the points that cartographers and non-cartographers selected as critical, (2) the selected points are located at places of high slope change, and (3) the fact that the same critical points were preserved on all three levels of generalization indicates their significance in determining the character of the line. In another study, George Jenks (1981) mentions that there are highly significant points in every line that define its geographic configuration. Subsequently, Jenks underlines their importance arguing that "a sparse but carefully selected set of sample points can be used to create a faithful representation of any line" (1981, 4). Jenks discerns two types of characteristic points: (1) significant economic, cultural, and political locations, the selection of which depends on the purpose and subject of the map; and (2) natural, important, or basic locations that relate to the structure of the line. These points provide the line's individual and distinctive form. Their location is related to changes in the slope (direction) or to large fluctuations of the line. In research focusing on the comparison of three simplification algorithms through mathematical, as well as empirical, criteria, Ellen White (1985) describes results similar to those of Marino's empirical study. Some important findings yielded from this research are the following: (1) responses from the subjects form a rating of detection points, considering some points to be more "critical" or more "important" than others because they are detected more systematically than others; and (2) points selected by cartographers and non-cartographers are in close agreement (White 1985). According to the results of White's study, the slight differences in the points selected by cartographers and those selected by non-cartographers are due to the perception of the observer and especially to the fact that cartographers tend to focus more on the individual characteristics of the lines.

In computer cartography, the concept of critical points forms the conceptual foundation of line simplification algorithms. Most algorithms analyse line structure using geometrical criteria (length, areal displacement, perpendicular distance, angular change, etc.), as well as tolerances, depending on the level of simplification and the purpose of the map. Line generalization research has long relied on Attneave's (1954) theory on critical points to quantitatively assess simplification procedures (see, e.g., McMaster 1987). The assessment of any line simplification algorithm can be carried out by using several quantitative criteria, such as the mathematical measures proposed by Robert McMaster (1986). However, there are still a few questions to be addressed: Are the two lines, before and after generalization, visually similar? Are the retained points really critical? Can the derived lines be assessed aesthetically?

In the process of manual line simplification, the cartographer examines the significance of each location with regard to global and local criteria. The cartographer estimates the information that every point contributes to the line's basic shape in relation to the level of simplification and the line's form. The complexity of this process makes automation very difficult. The difficulties are compounded by the manner in which vector data represent, register, handle, and depict continuous phenomena (such as lines on a map) in a digital environment, which tends to conflict with map readers' demands.

Every point located at a place of high slope change is not necessarily a critical point, and, likewise, critical points are not located only at those parts of the line. Barbara Buttenfield (1989) states that cartographic lines should be "handled" differently, during the simplification process, depending on their geomorphological nature and character. Lines can be divided into two categories (Buttenfield 1989): (1) those whose structure changes with scale (scale-dependent) and (2) those whose structure does not change (scale-invariant). If this categorization is accepted, a rule of thumb cannot be defined that deterministically retains or rejects points in a simplification process. In fact, most line simplification algorithms do not provide any options for preserving critical points, nor do they model or assess the visual quality of the results. White's (1985) research came to similar conclusions: less than half of the critical points identified by the subjects coincided with those detected by what she considers the most valid line simplification algorithm – the Douglas-Peucker algorithm (Douglas and Peucker 1973). Khagendra Thapa (1988a, 516) states that "some of the critical points which are likely to cause spikes in the generalized lines must be eliminated if the generalized lines are to be smooth, uncluttered, and aesthetically pleasing."

However, the very essence of many algorithms' structure precludes them from properly retaining the most characteristic points. The analysis of a line by repetitive use of a set, specific geometric criterion, leads to results that depend both on the criterion itself and upon the tolerances set by the user. Furthermore, global analysis of a line (typified by the Douglas-Peucker algorithm) can easily fail to recognize its character at a local level. Geoffrey Dutton (1999) underlines this fact, mentioning that the analysis of the line as a whole for the selection of the points that form its shape may lead to visually unacceptable or even erroneous results. This is because visual observation is affected by a combination of factors, such as the level of scale change, the complexity of the line, and the type of phenomenon represented. In conclusion, Dutton suggests that "by segmenting line features to be more homogenous, then applying appropriate algorithms and parameters to each regime individually, simplification results can always be improved" (1999, 36). Maheswari Visvalingam and Duncan Whyatt (1990) similarly express the concept in a comparison the Douglas-Peucker algorithm with their own:

> Points selected by the Douglas-Peucker algorithm are not always critical. Manual generalizations take into account the relative importance of features. This is partly dependent upon the purpose of the map. (224)

In computer science, especially in topics such as computer vision, pattern recognition, and signal processing, a large number of algorithms for detecting critical points have been developed, mainly to address the problems of line approximation, curve segmentation, and feature detection. The majority of these algorithms are based on curvature computation at each point of the curve by analysing angularity. Critical points considered are those located at the curvature's maxima and minima. According to the classification of critical point detection algorithms presented by Zhilin Li (1995), the majority of these belongs to the "corner detection" category. Most early algorithms approximated curvature based on computations of the angle θ – or its cosine ($\cos\theta$) – at each point I between two points of the curve ($\mathrm{I}-k$ and $\mathrm{I}+k$), with the k parameter set by the user (Li 1995).

The concept of the "region of support" around each point became a principle that would form the basis of many algorithms coming from the computer science domain. A precise determination of the support region is much more important than the chosen curvature measure. Based on this principle, and trying to avoid shortcomings from using any parameter, a non-parametric algorithm is developed (Teh and Chin 1989) in which the support region of each vertex is determined on the basis of local properties of the line. Secondarily, measures are used to estimate curvature.

To better handle the problem of noise, and especially to overcome the shortcoming of directly applying the mathematical definition of curvature to discrete representations of linear features, the original line should be smoothed by a filter (most commonly the Gaussian filter) before computing the curvature (Ansari and Huang 1991). According to Philippe Cornic (1997), this approach raises the problem of selecting the appropriate filter width, since a rather small Gaussian filter width may lead to insignificant detections whereas a large width may exclude certain critical points from detection. Thus, several researchers (Rattarangsi and Chin 1992; Pei and Lin 1992) suggest that the lines should be smoothed by the Gaussian filter at several levels (from minimum to maximum).

Following the concept of line analysis at a local level, new algorithms have been developed that are not based on the estimation of curvature for critical point detection. Cornic (1997) introduces a non-parametric algorithm that does not characterize each point by computing the curvature parametrically. Instead, it applies a region of support around each point of the line and rates the points close to the left or right limit of the region. Finally, the algorithm detects as critical points those gathering the higher scores. Terence Cronin (1999) introduces a similar algorithm, in which every point of the line is classified in one of 18 groups, with its position and orientation in relation to its predecessor and successor as criteria. In general, the points are encoded as convexity (local maxima), concavity (local minima), and run point (straight angle). Cronin's algorithm initially detects the maxima and minima, filters them by using an error budget procedure in order to discard sequences of obtuse vertices or shallow curvature sequences, and selects the critical points. Finally, a new method that uses wavelets to detect critical points has been developed (Antoine and others 1997).

Although the digital image processing domain applies the concept of critical points differently, and linear features are represented using raster data structures, its approximation techniques are very interesting. The formulation of methods of exclusive critical point detection in that domain sets precedents for similar efforts in cartography. Their main characteristic is setting a test area around each examined vertex. Within this area, the curvature of the line is estimated by analysing its angularity, and, according to Attneave's (1954) considerations, "crucial" locations can be detected. Thus, the examination of the points' significance is carried out at a local rather than a global level. This fact is a subject in need of further research by cartographers. In addition, the use of techniques for reducing the effects of discrete representation of continuous features in the computer environment by smoothing lines differentially through filters according to their local morphology is interesting. Finally, recent efforts (Cornic 1997, Cronin 1999) to formulate non-parametric algorithms that are independent of the user's subjective involutions and trial-and-error processes can be considered pioneering.

Aim of the Study

This article introduces a method based on geometric principles for detecting critical points using the length ratio (LR) as a measure of estimation for the slope change along the points defining a digital line. The LR index can be calculated for each point of the line and can then be assigned to each point. When the values of LR are scanned from one end point to the other, several fluctuations are observed, with local maximum values at those locations where the line is most different from a straight line. All vertices associated with LR values higher than a given threshold are considered critical points. In the special case of open lines, the two end points are also regarded as critical points. Three geomorphological lines from the study of Marino (1979) and the theoretical line from the study of Thapa (1987) were used to test the proposed method. Finally, the method was used to compare the results of two line simplification algorithms applied on five line simplification tasks on the coastline of Peristera Island, a small island located at the centre of the Aegean Sea and characterized by a high degree of shape complexity.

Description of the Method

In a digital environment, the proposed index is applied on lines with a vector structure, that is, a discrete number of points connected by vectors. The criterion of detecting critical points along a line is chosen in such a way that those vertices with high changes of line slope are located. The central idea is to clip the line around each vertex and determine its slope change independently, using line length as a geometric criterion. The LR method adapts the concept of "region of support," presented in many critical point detection algorithms originating from the field of computer science. The line is clipped by applying a circle centred at each vertex. The radius (R) of the circle is set before application. Consider points P1 and P2, which are defined as the two consecutive intersections of the line with the circle. It is possible to estimate the length L of the line along the path between two points and the chord length S between them (see Figure 1). The index of length

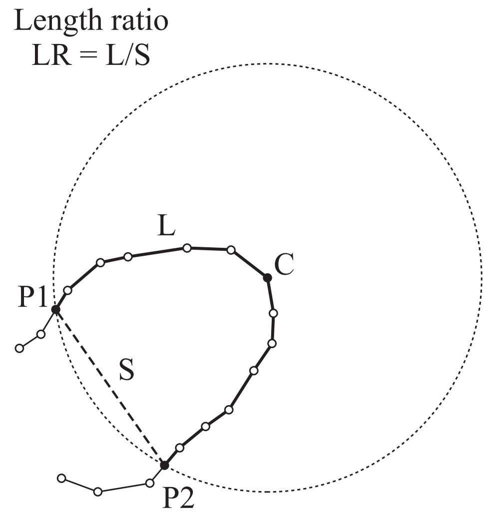

Figure 1. Typical case of intersection between the line and the circle.

ratio (LR) is defined as follows:

$$LR = \frac{L}{S}$$

Such a circle may

1. intersect the line at two points P1 and P2 from both sides of the vertex (Figure 1). In this case, which is the most usual one for the intended radii, the intersections P1 and P2 are detected. Then length L is calculated (as the sum of the intermediate rectilinear segments) as well as the length of chord S.
2. intersect the line at one point P1 (Figure 2a). This case can occur at the end locations of open lines. The length L between the circle's centre and the intersection point is calculated, and the chord length S is equal to the radius R.
3. intersect the line at more than two points P1, P2, P3, . . . , Pn from both sides of the vertex (Figure 2b). In this case, the closer intersections to both sides of the vertex P1 and P2 are detected. The procedure operates as in case 1.
4. intersect the line at more than two points P1, P2, P3, . . . , Pn at one side of the vertex (Figure 2c). In this case, the intersection closer to the vertex P1 is detected. The procedure operates as in case 2.
5. not intersect the line (Figure 2d). In this case, the LR index cannot be applied and a smaller radius is required.

Through the application of a circle with constant radius, visiting all line vertices one after the other, the line is equally clipped on the basis of a common measure. Thus, a constant test area is produced for each vertex in which the length L, the chord length S, and the LR index are calculated. The LR index is closely related to the

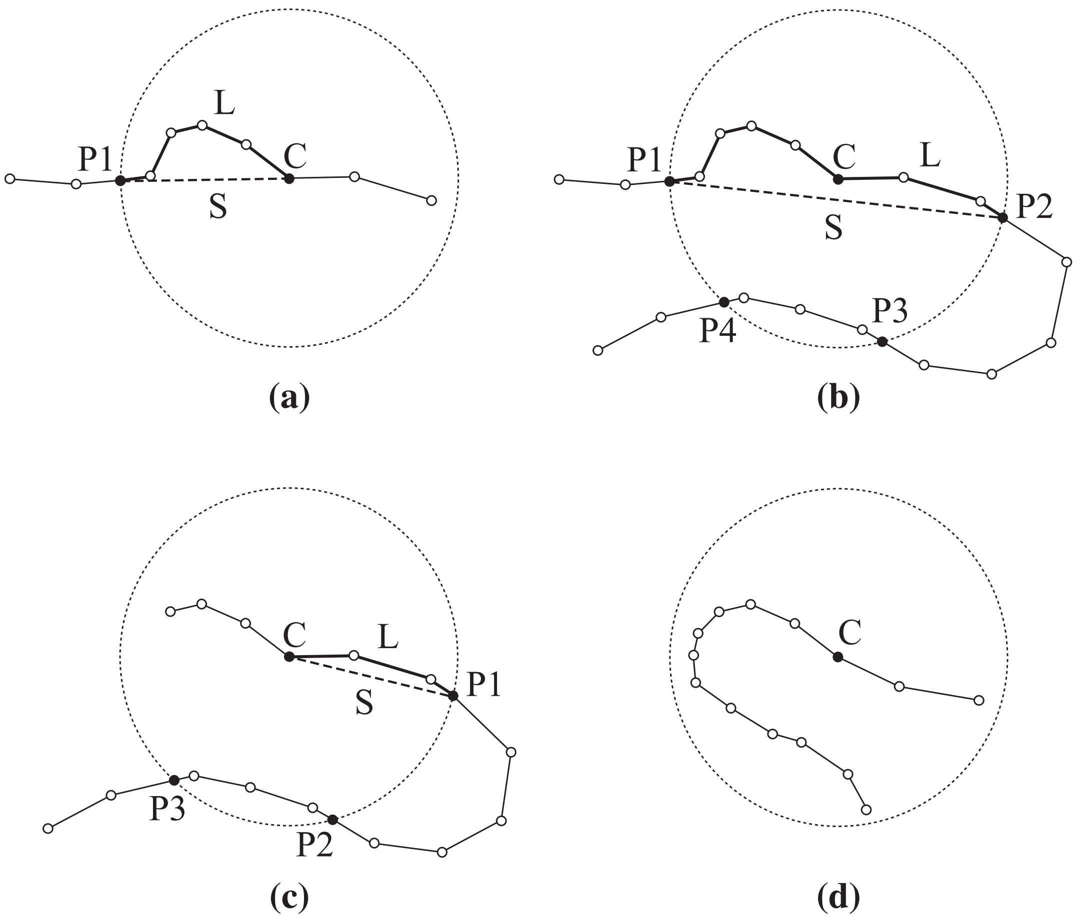

Figure 2. Four alternative cases of intersection between the line and the circle.

self-similar fractal dimension (Mandelbrot 1982) and thus expresses quantitatively the degree of complexity of the line section clipped by the circle.

Assuming that the digitization step of the line remains approximately constant (so as to measure lengths on a common basis) and is smaller than the circle radius, the length L and chord S depend on the shape of the line between the two points of intersection. The length L increases, while the chord length S tends to decrease, as the slope change of the line increases (see Figure 3). Consequently, the LR index varies with the slope change of the line; as slope change increases, the LR index increases as well. The LR index is dimensionless, and it is always greater than or equal to 1 (LR $>=$ 1). The equality corresponds to straight-line segments. Therefore, the LR index can be used as a measure of the slope change variation. The vertices of the line considered critical are defined as those associated with local maxima of LR values (see Figure 4).

The basic principle of the proposed method of critical point detection is the estimation of slope changes through

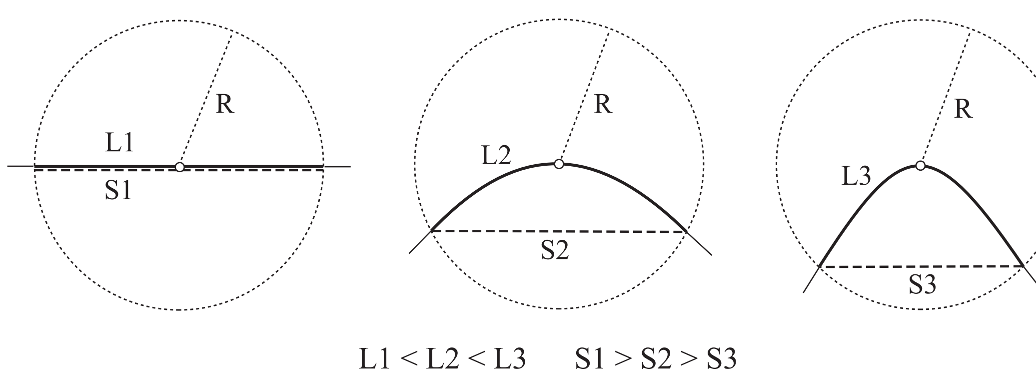

Figure 3. Variation of length L and chord S in relation to curvature.

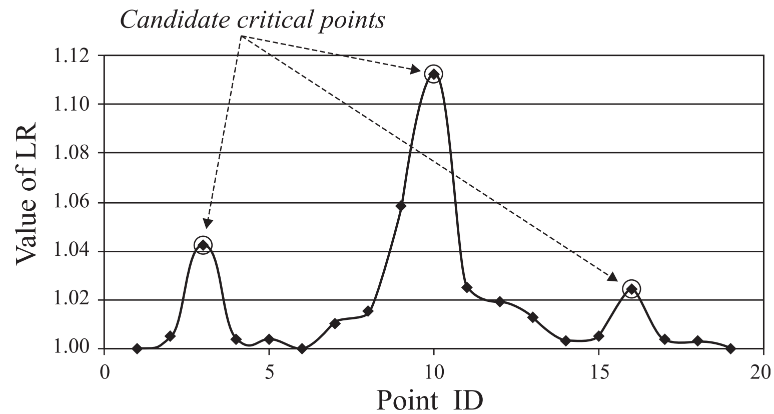

Figure 4. Candidate critical points in a diagram of LR values.

the LR index at the local level around each vertex. The size of the circle directly affects the quality of the results, as the circle determines the test area. If the radius is increased, we can observe that both length L and chord length S vary positively as long as the segment around the vertex expands. However, their variation is not equal and depends on the irregularity of the line. To illustrate the above, an example of the variation of length L and chord S over a wide range of radii (R) is presented in Figure 5, starting from a size equal to the average step of line digitization and advancing proportionally, for three characteristic types of points respectively (see Figure 6).

By interpreting Figures 5 (a), (b), and (c) we can observe that the rate of increase of length L relative to chord length S varies more at positions with high angularity. Specifically, the difference of length L with regard to that of chord S appears higher at high slope changes (Figure 5a); this difference decreases at moderate slope changes (Figure 5b) and converges to zero at straight sections (Figure 5c). Figure 7 illustrates the variation of percentage increase of LR values with respect to the range of different radii related to the average step size of the line, that is, the sum of the lengths of the linear segments representing the line divided by their number. When applying circles with a radius equal to two to four times the average step size, the maximum deviations between L and S appear; hence we have distinguishable values of LR for critical point detection. When a circle's radius exceeds the average step size by about five times, LR values tend to express global rather than local characteristics of the digitized line shape.

The values of LR can be classified into three groups (A, B, and C), on an ordinal scale, according to the line shape characteristics. After an empirical exploration, the following limits of LR values are selected to define the three groups:

- Group A (LR values ranging from 1.04 to 1.15): This group of critical points refers to locations of smooth slopes (up to 120°) with basis vs. height ratio between 4:1 and 11:1 (see Figure 8a).
- Group B (LR values ranging from 1.15 to 1.30): This group of critical points refers to locations of sharp slope changes (90°–120°) with basis vs. height ratio between 3:1 and 4:1 (see Figure 8a).
- Group C (LR values greater than 1.30): This group of critical points refers to locations of peaks with slopes less than 90° with basis vs. height ratio smaller than 3:1 (see Figure 8a).

It should be mentioned that according to the classification defined above, a lower limit (threshold) for LR values of 1.04 is used. Empirical analysis of several demo lines varying in width according to the standards of linear cartographic symbols and shape indicates that vertices associated to locations with bends having a basis vs. height ratio smaller than 11:1 can be considered critical. Application of the LR indexes shows that these locations have LR values higher than 1.04. This lower limit excludes from the set of critical points those vertices that are associated to minimum slopes, straight segments, or even "noise" inherent in any digital representation.

Dutton (1999) suggests that an important criterion for a sound selection of points in a line generalization process is the point-by-point estimation of local line sinuosity. For this reason, Dutton introduces the statistic "Measure of Sinuosity." For each point of the line, the ratio of distance along a digitized line between $\pm k$ adjacent points to the length of the trend line connecting these endpoints is calculated (see Dutton 1999, 41, Figure 4). The "Sinuosity Values" (SV) calculated by this method are dimensionless, real numbers and express the slope change of the line around each vertex. The number (k) of the adjacent points that "participate" in the calculation of the SV defines the width of the test area and therefore the values of the SV index. Dutton concludes that a more robust estimation of line sinuosity is obtained by calculating the average of the SVs resulting from sequential application of the SV index across a small range of adjacent points.

The structure of Dutton's Measure of Sinuosity is very similar to the LR structure. The two indexes estimate the

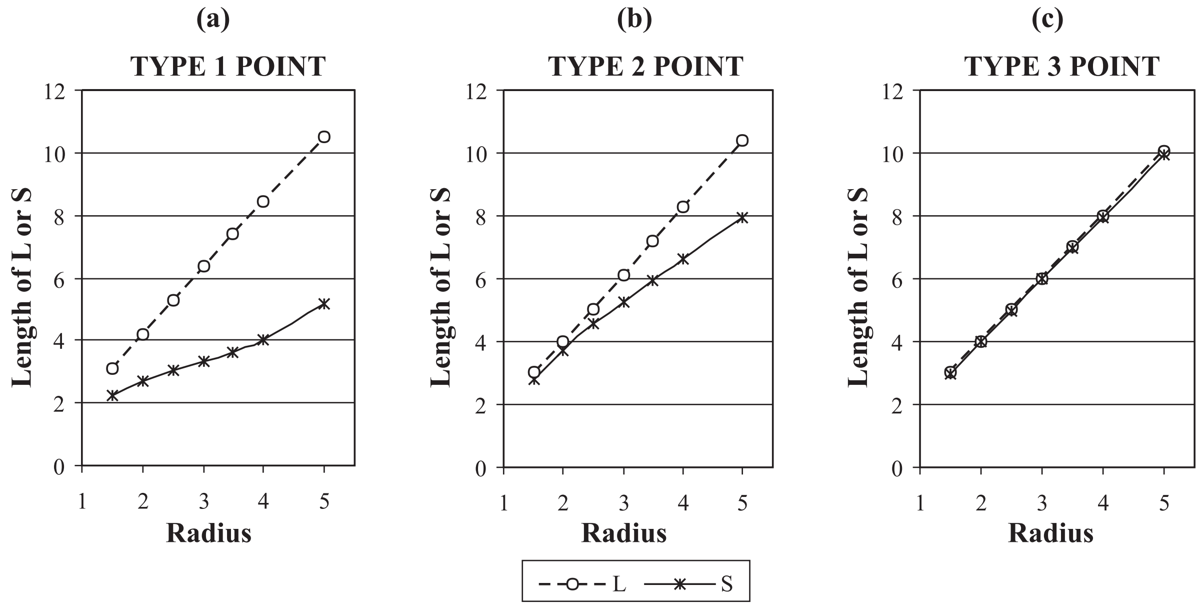

Figure 5. (a) L and S variation over a range of different radii at a location of sharp slope variation (point 1). (b) L and S variation over a range of different radii at a location of a curve peak (point 2). (c) L and S variation over a range of different radii at a location of straight section beginning (point 3).

irregularity of the line in a similar way. By applying a specific transform classifier, Dutton associates the SVs to three sinuosity levels (see Dutton 1999, 42, Figure 5). A comparison between the three groups of the LR values and Dutton's classification (Figure 8b) shows that SVs between 1.06 and 1.34 correspond to a line's "medium sinuosity" areas. These values are very close to the limits of group A and B LR values (1.04–1.15 and 1.15–1.30 respectively). Dutton indicates that SVs higher than 1.34 correspond to "high sinuosity" areas. This lower value is very close to the lower limit of LR values of group C (1.30). It is obvious that the LR values that correspond to the critical points of a line would coincide with Dutton's areas of medium and high sinuosity areas. The LR's lowest limit (1.04) occurs in the high levels of the "low sinuosity" area. In contrast, LR values do not exist in the remaining range of low sinuosity, as long as critical points are not detected in areas of low slope changes.

Finally, with the aim of estimating the irregularity of the line at local level, two kinds of LR indexes are defined:

- The Local LR (LLR) is calculated when a circle equal to two times the average step of digitization is applied at the line.
- The Average LR (ALR) is defined as the average of LR indexes when circles with a radius equal to the average step of digitization (R1), two times the average step of digitization (R2), three times the average step of digitization (R3), and four times the average step of digitization (R4) are sequentially applied at the line. Thus, ALR is expressed as follows:

$$ALR = \frac{LR_{R1} + LR_{R2} + LR_{R3} + LR_{R4}}{4}$$

The slope change of the line is estimated at a local level, around each vertex, by applying both indexes.

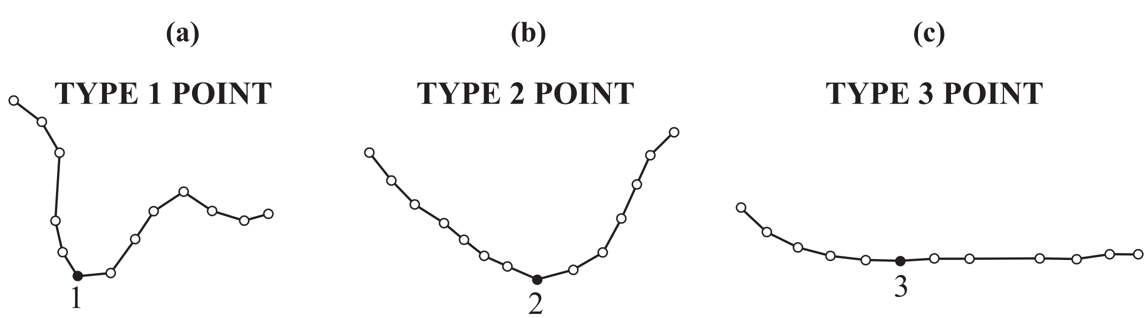

Figure 6. The three types of characteristic points (descriptions are given in the captions of Figures 5a, 5b, and 5c).

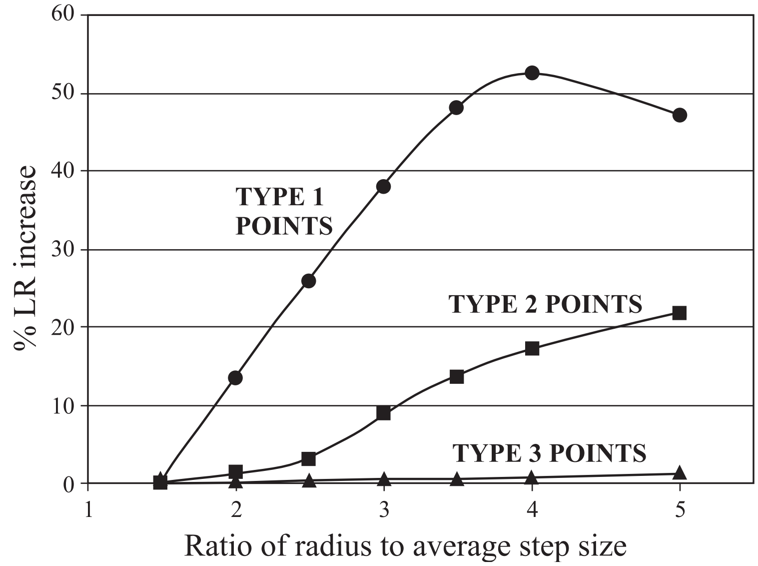

Figure 7. The percentage increase of the LR index over different radii at the three characteristic types of points.

The difference between the two indexes is the limit of the examined region. The slope change is estimated in a small region around each vertex by applying the LLR. As a result, the influence of the line's local attributes is high at the index. Subsequently, the ALR is applied, so that the slope change is estimated both at an extremely local level (R1) and at wider levels (R3, R4). Thus, the influence of the adjacent vertices' attributes decreases. The importance of each vertex is estimated in a larger range, and thus the values of ALR are more regularized than those of the LLR. In addition, they yield more precisely the importance of each point in the wider area of the line.

Comparison with Relevant Studies

The credibility of the results is estimated by applying the proposed method to lines with critical points known in advance. Lines depicting parts of the Mancos River, the Shenandoah River, and the Cape Argo coastline were chosen from Marino (1979); Thapa's theoretical line was chosen from Thapa (1987). The first three lines represent

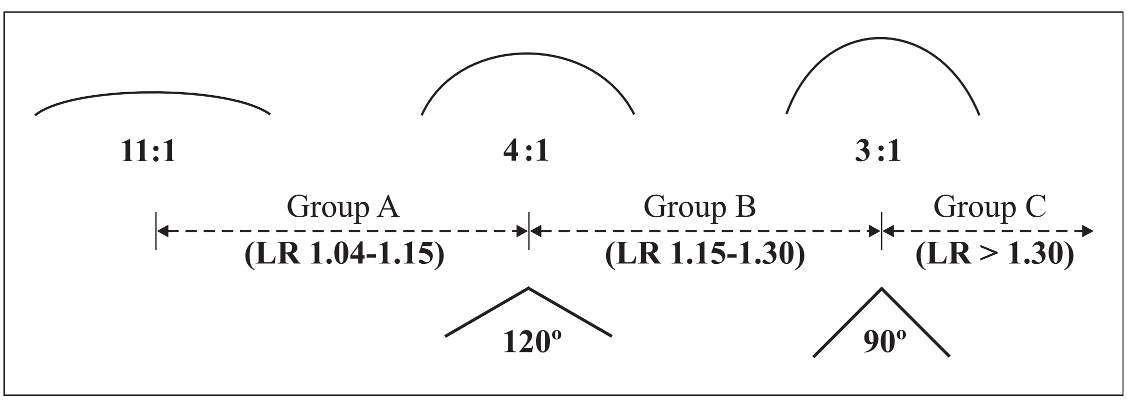

Figure 8a. Characteristic shapes of curves for the three groups (A, B, and C) of LR values.

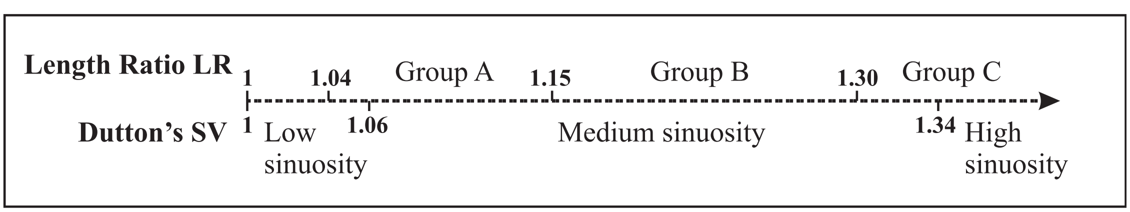

Figure 8b. Comparison between groups of LR values and Dutton's SV classification.

natural phenomena and constitute different geographical and geomorphological samples. The Shenandoah River is roughly sinusoidal, the Mancos River is characterized by high complexity, and the Cape Argo coastline includes both high-complexity and straight sections. The critical points were derived from the empirical study of Marino (1979). In the present study, the critical points related to the first level of simplification (Marino 1979) are used in order to include the largest possible sample. Thapa's (1987) theoretical line is a geometrical model, designed to represent a large assortment of shapes (spikes, sharp slopes, straight sections, continuous long curves, etc.) that lines can exhibit, either independently or in combination. The present study uses critical points produced by Thapa's mathematical model (1988b, 64).

The lines from studies by Marino (1979) and Thapa (1987) were chosen in order to check the results of the proposed method of detecting critical points directly against a study that reflects the way humans conceptualize and select critical points (Marino 1979) and a similar method based on mathematical assumptions (Thapa 1987). Thus, both of them constitute sound bases for checking the proposed method.

The raw data of the test lines were created by vectorizing scanned images from Marino (1979) and Thapa (1987) at a resolution of 300 dots per inch. The original lines were created after cleaning and smoothing the raw data, in order to be clear of redundant vertices, and smoothed following the standard cartographic procedure (Jenks 1981). The average step sizes of the test lines are presented in Table 1. Nevertheless, there are points of poor digitization due to the source quality. These points were marked and their LR values were checked in order to prevent them from affecting the final result.

The critical point detection was conducted by applying both LLR and ALR. The aim was to evaluate their functionality and to compare them on the basis of known facts. The parameters of implementation and the results are presented in Table 1.

Table 1 shows that the LR method of detecting critical points is in general agreement with the two relevant studies (Marino 1979; Thapa 1987). With the exception of the Cape Argo coastline, both LLR and ALR succeed in detecting 90% or more of the critical points noted in these two studies.

However, there are some cases of divergence in which the proposed method detects critical points that were not considered to be critical, mainly in Marino (1979); these are referred to as "extra points" in Table 1. This phenomenon is observed mainly in the Mancos River line. For example, the LR method detects small, isolated "breaks" of the line that Marino's study participants could not discern. On the other hand, Marino's study set an upper limit on the number of points participants could select. This limitation allows us to guess that some of the extra points would have been considered critical if participants had been permitted to select a greater or limitless number of points. For example, all the peaks of consecutive fluctuations in the line are detected by the LR method; in Marino's study, only some of these are selected. Considering Marino's study as a sound test, we conclude that some of the extra points are actually not crucial. They are located in low-significance positions, or are derived from a shortcoming in the LR index. However, some of the extra points could be considered as critical, comparing them with critical points located in similar line areas.

An examination of the total number of the critical points detected by the application of LLR and ALR yields important indications about the quality of the two indexes. At first glance, it is obvious that the success ratio of both indexes regarding the detection of the critical points presented in the studies of Marino (1979) and

Table 1. Parameters of LR index application and results of the comparison

Line	Thapa line	Shenandoah River	Mancos River	Cape Argo Coastline
Average step size*	0.44	0.22	0.18	0.30
Radius of LLR*	0.9	0.5	0.4	0.6
Radii of ALR*	0.45, 0.9, 1.35, 1.8	0.25, 0.5, 0.75, 1	0.2, 0.4, 0.6, 0.8	0.3, 0.6, 0.9, 1.2
Known C-P	45	53	40	53
LLR				
Detected	45	65	88	63
Common	41 (91%)	50 (94%)	38 (95%)	43 (81%)
Extra	4	15	50	20
ALR				
Detected	45	59	72	49
Common	40 (89%)	51 (96%)	35 (88%)	38 (72%)
Extra	5	8	37	11

*In mm on the map.

Thapa (1987) is almost the same (LLR gives slightly better results for three of the four lines). With the exception of a few cases, the critical points detected are common. The differentiation between the two indexes consists in the total number of points detected as critical. The number of critical points detected by the application of ALR is notably smaller (only in the Thapa line will both indexes detect an equal number of points). By examining the additional critical points detected in the application of LLR, we can observe that a large number of them correspond to local fluctuations of the line with minor importance for the retention of its shape. For example, many of these points are located in positions adjacent to other critical points or in positions of low or medium fluctuation preceding or following sudden ''breaks'' in the line. In these cases, the ALR values are more regular, that is, they do not have a local maximum but are increasing (or decreasing) to (or from) the local maximum value that corresponds to the point of greater slope change. The estimation of the slope change of a larger region around each vertex by applying ALR facilitates the detection of critical points located in isolated, low fluctuations in the line or in locations of low slope change. Finally, by using ALR, the critical points located in large, wide curves can be detected precisely at their peak (in many cases, LLR leads to the detection of points adjacent to the peaks). A more extensive and qualitative examination of the results with regard to the morphology of the mapped features yields the following observations:

THAPA'S THEORETICAL LINE

Because of the geometrical shape of Thapa's (1987) theoretical line, the gradation of LR values is in full accordance with the shape of the line, especially when examined globally. By examining the results in more detail, we can observe that critical points of Group A mainly correspond to small line breaks, smoothed slopes (up to 120°), and continuous small curves of the line. Group B consists mainly of continuous, large fluctuations and sharp slope changes. Finally, group C corresponds to locations of sharp breaks in slope, acute spikes, and medium and large fluctuations. The critical points not detected by the application of LLR and ALR are common and are located in positions of nearly zero slope change of the line. The extra critical points detected by application of ALR are located in small, narrow curves of the line (with a basis vs. height ratio equal to 5.5:1) and in peaks of small, continuous fluctuations that could be considered critical (Thapa's method detects the adjacent peaks). The same applies to the extra points detected by LLR, with the exception of one specific point, the detection of which is due to the index sensitivity in the local characteristics of the line. Nevertheless, it is important that the total number of critical points detected by the Thapa and LR methods is the same (45). The two mathematical methods have a satisfactory coincidence in detecting critical points.

THE SHENANDOAH RIVER LINE

For the sinuous Shenandoah River line, the success ratio in the detection of the critical points present in Marino's (1979) study verges on agreement by applying both LLR (50 of 53) and ALR (51 of 53) indexes. All the peaks of large curves and sudden ''breaks'' that characterize the line are detected. In fact, the LR values that correspond to these locations belong to groups B and C and constitute the majority. The rest of the LR values belong to group A, corresponding to smooth slopes and sudden small ''breaks'' in the line. The points not detected by LLR are located in solitary smooth curves and in positions of low slope change that form bends with a basis vs. height ratio greater than 11:1 (not detected by ALR either). ALR fails to detect another critical point located in a position following a narrow curve, since the index values near it are in a decreasing rate. Some points not considered critical in Marino's study are detected by LLR and ALR. Their number is not high, but they are of great interest. A small number of these extra points are located in positions of short, sudden ''breaks'' in the line; they correspond to low LR values (approximately 1.05) and were not easily perceived by the participants in Marino's study. The other extra points detected by LLR correspond to local line fluctuations. The majority are located in positions adjacent to critical points or in positions of minimum importance for the preservation of the line's shape. By applying ALR, we can eliminate all these points. The application of the LR method to the Shenandoah River line clearly indicates how estimating the irregularity in a range of areas around each vertex by using ALR leads to the elimination of many ''superfluous'' critical points detected by LLR. However, the Shenandoah River line reveals a rarely occurring shortcoming of ALR. Five of the eight extra points belong to group C; they are located in positions adjacent to large peaks of crucial importance. The problem is encountered when, for reasons relating to the shape of the curve, the area examined for R4 exceeds the ''local'' limits. Thus, the LR_{R4} index is very high and increases the value of ALR. Although this issue is rarely encountered, it requires more thorough research.

THE MANCOS RIVER LINE

For the Mancos River line, the majority of the points considered critical in Marino's (1979) study are detected by both LR indexes (38 out of 40 using LLR and 35 out of 40 using ALR). Those that were not found are located in parts of curves adjacent to detected critical points with lower basis vs. height ratios (giving higher LR values). In these areas the LR values do not have local maxima but are either increasing or decreasing. What characterizes the LR index's application to the Mancos River line is the detection of a large number of extra points, resulting from the shape of the specific line. The Mancos River is a line of high complexity, having continuous wide or narrow bends. The structure of the LR index leads to the detection of a high proportion of these (those having a basis vs. height ratio smaller than 11:1). However, in Marino's study about half of these are considered critical. With the exception of two, the majority of the known critical points correspond to LR values higher than 1.08. Most of the extra points belong to Group A and fewer to Groups B and C. In a line of high complexity, the limitation on the number of selected points imposed by Marino could be an easy explanation for the deviation in the number of critical points between the two methods. However, if we accept Marino's study as a sound basis for comparison, a distinct shortcoming of the LR method is revealed: the LR index is sensitive to high-complexity locations in the line. Application of LR to the Mancos River shows that only positions of high slope change must be considered crucial in similar cases. This explains the fact that the majority of the extra points correspond to the low values of Group A. This issue, however, requires more thorough research.

THE CAPE ARGO COASTLINE

For the Cape Argo coastline, LR values correspond directly to the line morphology. It is for this line that the lowest ratio of success is achieved (43 of 53 points using LLR and 38 of 53 using ALR) with respect to the detection of the critical points selected in Marino's (1979) study. This is due mainly to the shape of the line, which includes both high-complexity and straight sections. Many critical points presented in Marino's study are located in positions of low slope change or zero change; the LR values that correspond to these are below the threshold. In high-complexity areas, participants in Marino's study selected adjacent points as critical. A number of these are not detected by the LR index, since their values are either increasing or decreasing. Finally, the LR index detects certain points not considered critical in Marino's study. These extra points belong to all groups of LR values. They are located in small, smooth fluctuations of the line, in positions of abrupt slope change, or in continuous peaks (some of which are considered crucial in Marino's study). Some of the extra points detected by LLR derive from the sensibility of the index to the local attributes of the line.

It should be mentioned that the LR method was also applied to the raw (non-cleaned and unsmoothed) data. This experiment showed that the same critical points were detected. This is mainly because the parameters for the cleaning and smoothing procedure were chosen to be close to the digitization tolerance as well as to the lower limit of LR values.

Test Application

The LR index was applied to the coastline of Peristera Island, an outline characterized by a high degree of complexity. The coastline was digitized from a paper 1:50,000-scale map with an average step size of approximately 15 m on the ground (or 0.3 mm on the map). The raw data were cleaned of duplicate vertices, spikes, and switchbacks after a ''weeding'' process. They were smoothed in order to produce a working data set. Then the data were cleaned and smoothed with parameter values close to the digitization tolerance.

The critical point detection was conducted by applying the ALR index. The four radii applied were 15 m, 30 m, 45 m, and 60 m on the ground (or 0.3 mm, 0.6 mm, 0.9 mm, and 1.2 mm on the map); the lower limit (threshold) was set at 1.04. Using this method, 135 critical points were detected (5.6% of the original). Figure 9 illustrates the original coastline and the critical points detected.

By assessing the location of the 135 critical points, we observe that the selected critical points satisfy the basic principle of retaining the shape and the character of the line. Locations of high slope change or line breaks that are crucial to effectively represent the basic shape of the line were successfully detected. The results of the test application are also consistent with the concept of LR value grouping according to shape regimes. Generally, 87 of the 135 critical points belong to group A, 28 to group B, and 20 to group C.

Following a closer look at the detected critical points, several observations should be discussed that refer mainly to parts of the line with higher or lower concentrations of critical points. First, some high concentrations of critical points, such as those indicated with the letters D and E in Figure 9, are identified. It seems that the LR method leads to overestimation of critical points in regions of high line complexity. The ALR values corresponding to these locations do not belong to a specific LR value group. They depend on the shape of the line. In areas of low slope change (indicated with D in Figure 9), the ALR values belong principally to Group A and secondarily to group B. In positions of high slope change (indicated with E in Figure 9), on the other hand, the ALR values belong mainly to Groups B and C. The same behaviour is also observed in analysing the Mancos River line, where several vertices associated with local maxima of the LR values were not chosen as critical in Marino's empirical study. This shortcoming can be explained by the fact that the length L, and hence the LR measure, increases in proportion with the complexity of the line.

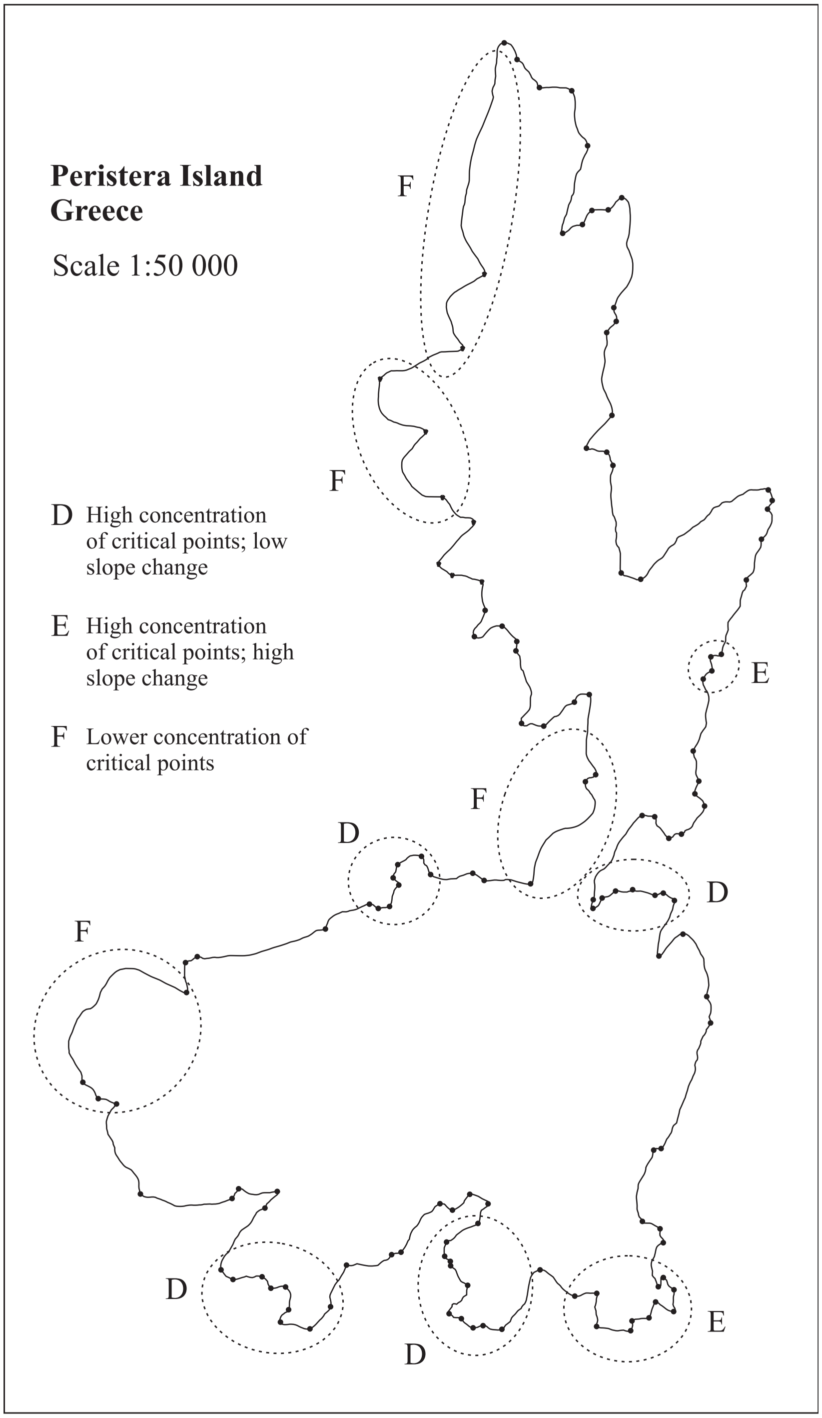

Figure 9. The outline of the Peristera Island coastline, digitized from a 1:50,000 map, and the 135 critical points symbolized by black dots.

It seems that the LR index is more sensitive in such situations.

The second observation concerns parts of the line with a lower concentration of critical points, such as those indicated with F in Figure 9. These cases consist of long, "wide" peninsulas with constant slope change. The LR values corresponding to the vertices that define their shape are constant (or have very low fluctuations) and fall below the threshold value. The LR method is structured in such a way as to detect positions of high slope change. According to Attneave (1954), critical points appear in these locations. Thus, the constant slope change peninsulas presented in the Peristera Island coastline do not present any critical points.

Comparison of Two Line Simplification Algorithms

The Peristera Island coastline was generalized with two line simplification algorithms, *pointremove* and *bendsimplify*, using ESRI's Arc/Info v.8.1 software platform. *Pointremove* is based on the algorithm developed by David Douglas and Thomas Peucker (1973) with some enhancements. Douglas and Peucker introduced an operator that eliminates the redundant points (detail) derived from a digitization process so as to produce sufficient abstraction of a line. The algorithm is structured so as to retain the points located in the slope change areas of a line and remove all other points. *Bendsimplify* is a further development of the key idea presented by Visvalingam and Whyatt (1993) and is based on research developed by Zeshen Wang and Jean-Claude Müller (1998). This line simplification operator aims to retain the curved parts of a line. It is based on the detection of the bends of a line, the analysis of their attributes, and the elimination of the insignificant bends on the basis of their attributes. These two algorithms were compared on the basis of the points preserved after their application in relation to the critical points detected on the original coastline. A qualitative and quantitative comparison of the retained vertices is discussed in this section. The generalization tasks include the simplification of the coastline at five different levels using each algorithm. The resulting coastlines are presented at scales 1:100,000, 1:250,000, 1:500,000, 1:1,000,000, and 1:2,000,000 respectively. In each case, the number of retained vertices is defined according to "principles of selection" (Töpfer and Pillewizer 1966), which in the case of coastlines is expressed as follows:

$$n = n_0 \frac{S}{S_0},$$

where S_0 and n_0 are the scale and the number of vertices of the original map and S and n are those of the derived map. The number of points required to represent a generalized line does not always vary linearly with the scale. Nevertheless, when the line and the reduction ratio are specified, the "principles of selection" can be used as an acceptable way to estimate the features of the derived line. In the present application, the "principles of selection" are used as a general accepted cartographic rule with the sole purpose of handling the tolerance values and defining the lines for the target scales. The tolerances are defined to equalize the number of preserved vertices at each level. Table 2 illustrates the parameters of the five levels of line simplification performed; Figure 10 portrays the simplified coastlines.

Table 3 presents the number of common points sharing each level of simplification with the set of the detected critical points on the original line of scale 1:50,000. For the first level of simplification, where the number of the retained points is high (50% of the original), the *pointremove* algorithm preserves all the critical points while *bendsimplify* preserves 84%. When the simplification level is increased, the *bendsimplify* algorithm percentage greatly decreases. On the second level, *bendsimplify* retains less than half of the critical points (49%). The fourth level of simplification is interesting, since the number of vertices shaping the simplified lines (129) is approximately equal to the number of critical points (135). It is observed that *bendsimplify* preserves only 15% of the critical points. On the fifth level, where the vertices shaping the lines are fewer in number than the critical points detected on the initial line, *bendsimplify* retains the least critical points. On the other hand, the *pointremove* algorithm detects almost all the critical points on the second level and more than 80% on the third level. On the fourth level, it retains 54% of the critical points. Even on the last level, the *pointremove* algorithm has a high ratio of success (35 critical points out of 61 line vertices). Similar results can be found in White's (1985) research, which assessed several line simplification algorithms. In assessing the validity of both algorithms in absolute numbers, one might conclude at first glance that the results of the *bendsimplify* algorithm have many shortcomings, since a significant number of the points selected by its application do not coincide with the predefined critical points. But which of the critical points does *bendsimplify* eliminate? And are these points of major importance for the preservation of the basic shape of the coastline?

Table 2. Retained vertices of Peristera Island Coastline for the five simplification tasks after applying "principles of selection"

Level	Nominal map scale	Retained vertices
1	1:100,000	1207 (50%)
2	1:250,000	485 (20%)
3	1:500,000	246 (10%)
4	1:1,000,000	129 (5%)
5	1:2,000,000	61 (2.5%)

The set of vertices preserved by the two simplification algorithms that are members of the set of critical points detected on the original line are classified into three groups (A, B, and C) of LR values. The results are presented in Table 4 for the five simplification tasks. Table 4 shows that the majority of the points selected by the *bendsimplify* algorithm correspond to Group A, whereas vertices corresponding to medium or high LR values (Groups B and C) are rarely selected. The *bendsimplify* algorithm does not retain points of high LR values after the second simplification level. In both

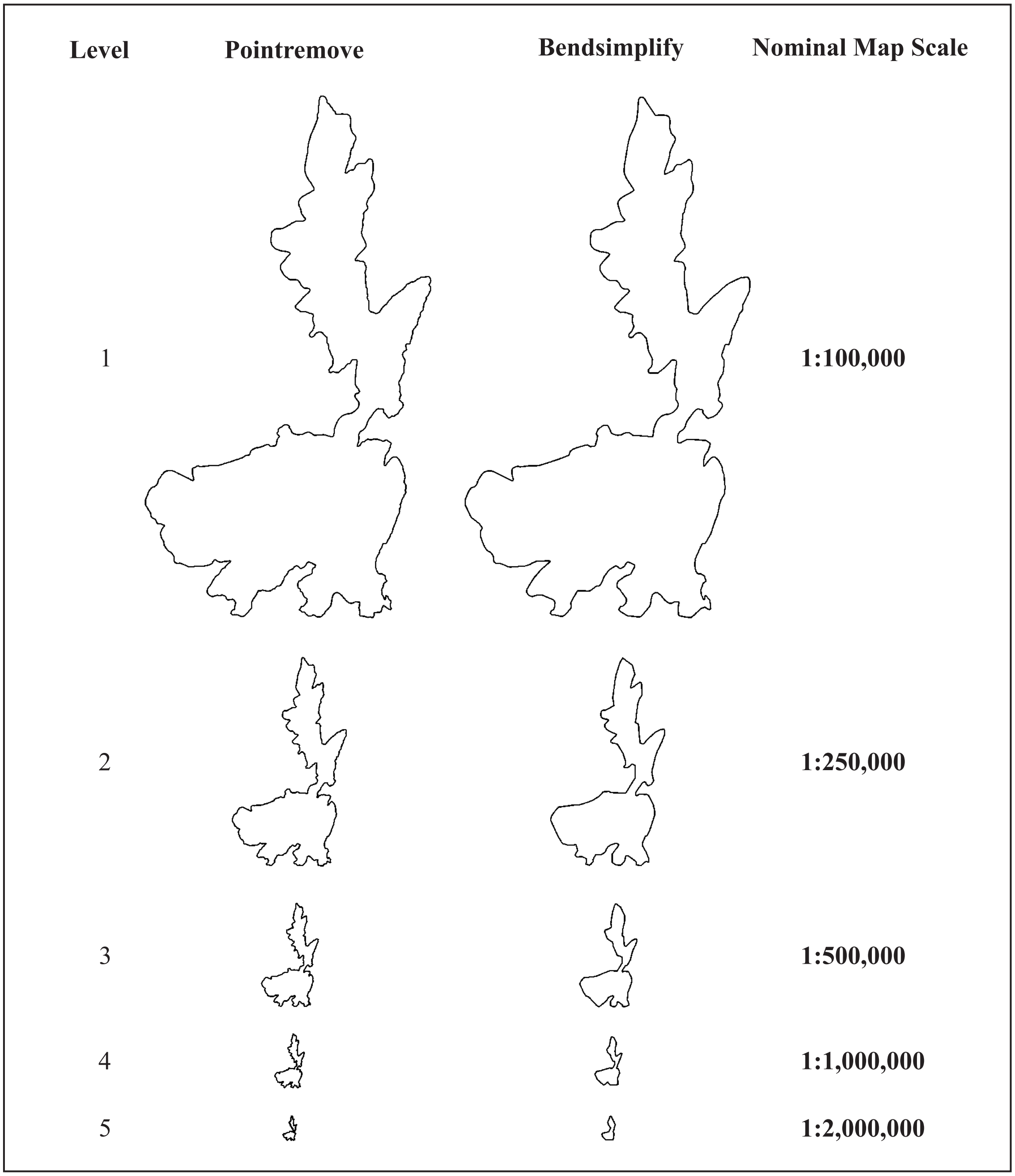

Figure 10. Simplified lines of the Peristera Island coastline at five derived scales.

Table 3. Retained critical points after the five simplification tasks

	Retained critical points	
Level	Pointremove	Bendsimplify
1	135 (100%)	113 (84%)
2	130 (96%)	66 (49%)
3	114 (84%)	39 (29%)
4	73 (54%)	20 (15%)
5	35 (26%)	8 (6%)

cases, more than 70% of the preserved vertices belong to Group A. In fact, the majority of the values in the first group (A) fluctuate at fairly low values (from 1.04 to 1.08). This implies that the retained vertices represent areas of smooth slope changes (these areas usually define the end points and the peaks of curves with large fluctuations) that must be preserved in order to maintain the shape of the line. The simplified lines appear smoothed to a large extent. At the first two levels of simplification, the *bendsimplify* algorithm retains all large, wide curves (see Figure 10 and areas indicated with F in

Table 4. Retained critical points classified into the three groups of LR values after the five simplification tasks

	Group A		Group B		Group C	
Level	*Pointremove*	*Bendimplify*	*Pointremove*	*Bendsimplify*	*Pointremove*	*Bendsimplify*
1	87	75	28	20	20	18
2	83	46	28	13	19	7
3	69	29	26	7	19	3
4	37	14	17	5	19	1
5	15	6	6	1	14	1
Original line	87		28		20	

Figure 9). The positions of high complexity corresponding to low LR values (see Figure 10 and areas indicated with D in Figure 10) are preserved, whereas positions corresponding high LR values (see Figure 10 and areas indicated with E in Figure 9) are not. At subsequent simplification levels, the high-complexity positions are removed. Simultaneously, only very large, wide curves are shaped (some merged with adjacent ones). Some narrow bays and peninsulas are not retained. At the last two simplification levels, the shape of the coastline is approximated (resembling an outline around the outer points of the line).

The *pointremove* algorithm preserves the majority of critical points at the first two simplification levels. Furthermore, the derived lines have the same distribution of critical points over the three groups as the original. Table 4 shows that the largest proportion of the retained critical points fall into Group A, so that the characteristic slopes of the examined line are preserved. By increasing the level of simplification where the number of critical points decreases, the percentage of critical points belonging in Group A decreases while the percentage of critical points belonging in Groups B and C increases. This shows that locations with smooth slope changes are not retained, in contrast to the high slope changes of the line. At all levels, the *pointremove* algorithm retains the large, wide curves (see Figure 10 and areas indicated with F in Figure 9). At the last two levels, however, these curves are shaped with the minimum number of points, resulting in a spiky outline. The complexity of the coastline is preserved at all levels, with a slight decrease only at the fifth level.

By comparing the two line simplification algorithms on the basis of the critical points detected by the LR index, we can observe many differences. This offers a rationale for examining differences between the two algorithms with regard to their structure and concept. In addition, know the attributes of the critical points enables us to acccomplish a quantitative assessment of the differentiations. The critical points selected by the two algorithms at each level of simplification and their classification in the three groups of LR values may allow us to make a quantitative analysis of what is perceived by the human eye.

The retention of high-complexity areas of the line by the *pointremove* algorithm can be considered a shortcoming, especially at high levels of simplification. The retention of the large, wide curves at small scales could be considered as preservation of detail. The retention of points with high LR values creates a spiky line shape, particularly at high levels of simplification. In contrast, the *bendsimplify* algorithm reduces the level of detail according to the simplification level. In addition, it minimizes the complexity of the line but preserves its basic shape, since the majority of the critical points detected on the original line that are ultimately retained belong to Group A. Thus, summarizing differences between the two algorithms with respect to the LR method of critical point detection and classification, we arrive at the conclusion that *bendsimplify* can be considered cartographically more appropriate than *pointremove*, especially for the depiction of lines at small scales. The lines derived after the application of the *bendsimplify* algorithm can be considered visually more aesthetic.

Concluding Remarks

The concept of critical points, presented by Attneave (1954) as characterizing line drawings and subsequently adopted by cartographers, has long guided research in cartographic generalization. It is significant indirectly as a guideline for the assessment of simplification algorithms and directly because it often serves as their objective, given that most simplification procedures depend, to a certain degree, on the retention or elimination of critical points.

The method proposed here is tailored to cover the needs of digital representation of lines in vector environments, as opposed to Thapa's (1987) method, which is appropriate only for digital lines in raster environments. Although the theoretical background of critical points guides line simplification by selecting points to preserve or to eliminate, the methods of automation present many differences.

The LR method is based on examining lines in neighbourhoods around each vertex. The method incorporates the idea of a "region of support" that characterizes analogous algorithms originating from the field of image processing. Neighbourhoods are identified by a circle of a defined radius centred at each vertex. Thus, the slope change is not estimated directly, by measuring the angularity; instead, it is calculated indirectly, through geometrical analysis of the line with length as the criterion.

The length of a line is the feature that depends on both the morphology of the line and its sampling (Buttenfield 1985). In the method proposed here, it is estimated that by setting the latter as "invariant," the former can be evaluated through the proposed index. Finally, the LR method detects critical points in a manner consistent with Attneave's (1954) considerations. The LR index is dimensionless and highly sensitive to irregularity variations, since the length L increases and the chord length S decreases proportionally to such variations.

The examination of each vertex independently results in an estimation of the local significance of each location. The procedure effectively simulates the perceptual (manual) process in which the significance of each location is estimated both locally and globally. This fact is very important for the analysis of cartographic representations, where the represented level of detail differs in proportion to the local morphology of the line. Researchers such as Dutton (1999), Visvalingam and Whyatt (1990), Buttenfield (1985, 1989), and Thapa (1988a) have also expressed this view.

In practice, users of the LR method must define two parameters: the radius (R) and the threshold of LR values to be retained. At present, we suggest that the radius be defined in proportion to the average step size of the digitized line and that the threshold of LR values be set at 1.04, a value that excludes from detection any curves that are longer than they are high by a ratio of at least 11:1.

The application of the LR index leads to results that are comparable quantitatively to relevant studies such as those of Marino (1979) and Thapa (1987). However, the undesirable sensitivity of the LR index in proportion to line complexity, as discussed in the previous sections, is a subject for future research.

The LR index does not detect points in long, wide curves with constant slope change. This is not a shortcoming of the method. These "silent" points that shape curves of low slope change may be of importance to the line simplification process, but they are not considered critical points and therefore are not detected by the LR method. Because this method was not developed to perform line simplification but to detect critical points, its functionality for this purpose can be considered satisfactory.

Finally, the LR method may facilitate research in cartography, particularly in line simplification, line segmentation, and multi-scale line representation. With respect to the topic of line simplification, cartographic research is focused on the development of an automated generalization procedure. The LR method might be incorporated in an automated line simplification process. Furthermore, the method may be used to segment line features in parts of homogenous characteristics in complexity or uniform characteristics in shape.

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